Chromatic Stable Homotopy
a course by Jacob Lurie

Lecture Notes by Christopher J. Schommer-Pries

Contents

1 How These Notes are Coming into Being

In the Spring of 2010 at Harvard University, Jacob Lurie taught "Math 252x - Chromatic Homotopy", a course on the chromatic picture of stable homotopy theory. The course description promised to begin with Quillen's work on cohomology theories and formal group laws and culminate with the resolution of the Ravenel conjectures via the work of Devinatz-Hopkins-Smith.

Chris Schommer-Pries LaTeXed these notes during class. They are currently in the process of creation, and so should be rapidly changing and evolving. They should be available at `http://sites.google.com/site/chrisschommerpriesmath/Home`. Any questions or comments can be given to me directly of by email (schommerpries.chris.math@gmail.com).

– When something doesn't make sense to me, I mark it with three big, eye-catching stars [[★★★ like this]]. If you can clear any of these up for me, let me know.

– If you have notes that I'm missing or if you have a correct/clear explanation for something which is incorrect/unclear, let me know (either tell me what you'd like to modify, give me some notes to go on, or update the tex yourself and send me a copy). Real (mathematical) errors should be fixed because it would be immoral to let them propagate (er . . . that is, sit there), and typographical errors hardly take any time to fix, so you shouldn't be shy about telling me about them.

Alternative lecture notes (written by Jacob Lurie) should also be available from the course website `http://www.math.harvard.edu/~lurie/252x.html`. Finally, I would like to thank Anton Geraschenko for his numerous inspiring lecture notes and for his source files, without which these notes would not be possible.

Acknowledgements

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2 Overview

Algebraic topology is the study of spaces through the assignment of invariants, but there are two opposing pressures: the power of the invariants v.s. the computability of the invariants. For example on one end of the spectrum the space itself can be viewed as a (complete) invariant. This is silly though. In the other extreme we have ordinary homology, which is easily computed. The topic of this course is going to be something in the middle, which balances these two pressures.

Let E be a multiplicative cohomology theory. That is to say we have

– Functors $E^n : \mathsf{Top}^{op} \to \mathsf{Ab}$ from spaces to abelian groups, and
– there are Mayer-Vietoris long exact sequences, satisfying the Eilenberg Steenrod axioms (except the dimension axiom)
– and $E^*(X) := \oplus E^n(X)$ is a graded commutative ring (this is the multiplicative property).

Here are some examples.

1. $E^n(X) = H^n(X; R)$ for a commutative ring R. This is ordinary cohomology.
2. $E^n(X) = K^n(X)$, the complex K-theory of X. For compact spaces X, K^0 is the Grothendieck group of complex vector bundles on X. There is a similar definition for higher groups, and also a way to extend to non-compact spaces.

For any multiplicative cohomology theory there is a spectral sequence,

$$H^p(X; E^q(pt)) \Rightarrow E^{p+q}(X),$$

known as the Atiyah-Hirzebruch spectral sequence (AHSS) which aids in computation. As with anything that applies to all spectra, this can often be difficult to compute. However sometimes it degenerates and we can compute it. For example, when $X = \mathbb{CP}^\infty$, we know it's cohomology very well. We have $H^*(\mathbb{CP}^\infty; \mathbb{Z}) = \mathbb{Z}[[t]]$, and by the universal coefficients theorem, any coefficients are given by tensoring with this. If the AHSS degenerates, then we have $E^*(\mathbb{CP}^\infty) \cong E^*(pt)[[t]]$ with $|t| = 2$.

Digression 2.1. Let's digress for a minute on the difference between formal power series and polynomial series. What do we really mean when we write "$H^*(\mathbb{CP}^\infty; \mathbb{Z})$"? What is given to us by the cohomology theory are the individual groups $H^n(\mathbb{CP}^\infty; \mathbb{Z})$. The object $H^*(\mathbb{CP}^\infty; \mathbb{Z})$ should be thought of as a *graded* object. It is really a matter of convention whether we choose to write $H^*(\mathbb{CP}^\infty; \mathbb{Z}) = \mathbb{Z}[[t]]$, or $H^*(\mathbb{CP}^\infty; \mathbb{Z}) = \mathbb{Z}[t]$. Both equations mean that $H^{2k}(\mathbb{CP}^\infty; \mathbb{Z}) \cong \mathbb{Z}$ for $k \geq 0$ with the basis element t^k.

However there is a difference between the expressions $E^*(pt)[[t]]$ and $E^*(pt)[t]$ (with $|t| = 2$). For example, in degree zero the later means $\oplus_{k=0}^\infty E^{-2k}(pt) \cdot t^k$, while the former means $\prod_{k=0}^\infty E^{-2k}(pt) \cdot t^k$. For example since the AHSS of $\mathbb{CP}^\infty$ degenerates for complex K-theory, we have $K^*(\mathbb{CP}^\infty) \cong K^*(pt)[[t]]$, and so $K^0(\mathbb{CP}^\infty) \cong \mathbb{Z}[[\beta t]] \cong \prod_{k=0}^\infty K^{-2k}(pt) \cdot t^k$. (Here β is the Bott element with $|\beta| = -2$). ◇

Definition 2.2. A multiplicative cohomology theory E is *complex orientable* if the AHSS for $E^*(\mathbb{CP}^\infty)$ degenerates. ◇

Here are some examples of complex orientable cohomology theories.

1. Ordinary cohomology
2. Any even cohomology theories, e.g. complex K-Theory (and also the previous example).

A choice of isomorphism $E^*(\mathbb{CP}^\infty) \cong E^*(pt)[[t]]$ gives a multiplicative generator $t \in E^2(\mathbb{CP}^\infty)$. When we are supplied with such an isomorphism, we say E is *oriented*. In ordinary cohomology this generator is $c_1 = t \in H^2(\mathbb{CP}^\infty; \mathbb{Z})$ and defines the first Chern class. More generally it gives Chern classes for the cohomology theory E. For K-theory the Chern class (which now, for 'technical reasons', is defined to be in degree zero) is given by $[L] - 1$ and there is a universal formula:

$$c_1(L \otimes L') = c_1(L)c_1(L') + c_1(L) + c_1(L')$$

The space $\mathbb{CP}^\infty \times \mathbb{CP}^\infty$ classifies pairs of line bundles, and again the AHSS degenerates and we get that the cohomology is $E^*(pt)[[u, v]]$. In addition to the defining line bundles (induced by the projection maps) there is also the line bundle which is the tensor product of these defining line bundles on $\mathbb{CP}^\infty \times \mathbb{CP}^\infty$. It's Chern class gives a power series $f(u, v) \in E^*(pt)[[u, v]]$. In our two favorite examples we have,

1. Ordinary: $f(u, v) = u + v$.
2. K-theory: $f(u, v) = u + v + uv$.

The tensor product of line bundles is commutative and associative (up to isomorphism) and this is reflected in the power series $f(u, v)$. It must satisfy the following properties:

– $f(x, y) = f(y, x)$ (symmetric),
– $f(f(x, y), z) = f(x, f(y, z))$ (associative),
– $f(x, 0) = f(0, x) = x$ (normalized).

This can be connected to the usual algebraic geometry of *commutative* rings by considering the even part of E. Let $R = E^{even}(pt)$. Then R is a commutative ring, and f is a power series $f \in R[[u, v]]$. The above conditions say that f is a (commutative 1-dimensional) formal group law over R. In this course we will only be dealing with commutative 1-dimensional formal group laws, and so we will often just say 'formal group law' and leave the 'commutative 1-dimensional' as implicit.

What we have is a construction, which is due to Quillen:

$$\{\mathbb{C} - \text{oriented cohom.} E\} \to \{(R, f \in R[[u, v]]) \mid f \text{ is a FGL}\}$$

This story is still in the realm of Algebraic Topology. We are assigning algebraic invariants to topological things like spectra. In may cases this turns out to be a compete invariant.

Remark 2.3. There is a "universal" example, which we will discuss at length in this course. This the the cohomology theory of complex cobordism MU. The ring $MU^*(pt)$ is the cobordism ring of stably almost complex manifolds. Milnor computed the complex cobordism of a point, $MU^*(pt) = L = \mathbb{Z}[u_1, u_2, \dots]$. There exist Chern classes for MU and hence a formal group law $f(u, v) \in L[[u, v]]$. ◇

Theorem 2.4 (Quillen). *L is the Lazard ring, and f is a universal formal group law.*

Another way to say this is that for all formal group laws (FGL) (R, g), there exists a homomorphism $(L, f) \to (R, g)$, which carries $f(u, v)$ to $g(u, v)$. Quillen knew that Lazard's ring was a polynomial ring on infinitely many generators, and $MU^*(pt)$ was too. He realized that they match up, via this construction.

Question 2.5. Are there any restrictions on the formal group laws that can appear?

The answer is no. Quillen's theorem says that in some sense the most complicated FGL (the one for MU) comes from topology. So it is unreasonable to think there are any restrictions on how complicated the formal group laws can be which come from topology. Suppose that we start with (R, f). This is determined by a map $MU^*(pt) = L \to R$. We can try to cook up a complex orientable cohomology theory by the following formula:

$$E^*(X) := MU^*(X) \otimes_L R$$

Why should this be a cohomology theory? Why should it have the necessary Mayer-Vietoris sequences? (It does if R is a *flat* L-module, but we'll see more general criteria later). We have the following vaguely stated theorem which we will explore later.

Theorem 2.6 (Landweber)**.** *This often works.*

Example 2.7 (K-theory)**.** The AHSS shows that complex K-theory of $\mathbb{CP}^\infty$ is $K^*(\mathbb{CP}^\infty) \cong K^*(pt)[[t]]$ a power series ring over $K^*(pt) = \mathbb{Z}[\beta, \beta^{-1}]$ with the generator having degree $|t| = 2$. Here we use the standard convention that the Bott element $\beta \in K^{-2}(pt) \cong \pi_2(K) \cong \pi_2(BU)$ has cohomological degree -2. This choice yields a theory with Chern classes in degree two. The formula for a standard such choice is $t = \beta^{-1}([L] - 1)$.

However there is a subtlety which appears for K-theory. Since $K^*(pt)$ is *2-periodic*, we can choose other isomorphisms $K^*(\mathbb{CP}^\infty) \cong K^*(pt)[[x]]$ with the degree of x any even integer we wish. Each choice gives a K-theory characteristic class for complex line bundles with degree $|x|$. If we use the isomorphism $K^*(\mathbb{CP}^\infty) \cong K^*(pt)[[x]]$ to identify $K^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \cong K^*(pt)[[u, v]]$, then we get a FGL which has the property that $f(u, v)$ has degree $|x|$ if u and v both have degree $|x|$. Below is a table summarizing some standard conventions. Here L is the universal line bundle, viewed as an element in $K^0(\mathbb{CP}^\infty)$.

$\lvert x\rvert$	Chern Class x	FGL
2	$\beta^{-1}([L] - 1)$	$f^\beta(u, v) = u + v + \beta uv$
0	$[L] - 1$	$f(u, v) = u + v + uv$
-2	$\beta([L] - 1)$	$f^{\beta^{-1}}(u, v) = u + v + \beta^{-1}uv$

Since each choice gives a formal group law, each choice also induces a homomorphism $L \to K^*(pt) = \mathbb{Z}[\beta, \beta^{-1}]$. We then (almost) get a cohomology theory by the assignment,

$$E^*(X) = MU^*(X) \otimes_L K^*(pt).$$

There are two subtleties. First, we won't get a $\mathbb{Z}$-graded theory unless the homomorphism $MU^*(pt) = L \to K^*(pt)$ is a *graded* homomorphism. This happens exactly for the choices with $|x| = 2$, e.g. $x = \beta^{-1}([L] - 1)$. For general x we only obtain a $\mathbb{Z}/2\mathbb{Z}$-graded theory. This is not such a big deal, as the ground ring $K^*(pt)$ is 2-periodic anyway. As we will see from the next section, each of these FGL are isomorphic (since β is a unit), and hence all of these ($\mathbb{Z}/2\mathbb{Z}$-graded) theories are isomorphic.

The second subtlety is more serious. The problem is that $K^*(pt)$ is *not flat* as an L-module, see Exercise B.4. [[★★★ Nevertheless $E^*(X) = MU^*(X) \otimes_L K^*(pt)$ is still supposed to define a cohomology theory which is isomorphic to K-theory????]] ⋄

How should we picture L? Geometrically we have $\operatorname{Spec} L = \mathbb{A}^\infty$ is the infinite dimensional affine space. The ring L corepresents formal group laws. However formal group laws can be isomorphic. Any other choice of generator $t \in E^*(\mathbb{CP}^\infty)$ is given (up to scaling by units) by $t \mapsto t + a_1 t^2 + a_2 t^3 + \cdots$. We are really interested, not in choosing a particular orientation, but just in the orientable theories themselves. So we really want $\operatorname{Spec} L/G$, where G is the group of corrdinate changes. Define the stack $\mathcal{M} = [\operatorname{Spec} L/G]$. This is the moduli stack of formal groups.

Idea 2.8. The geometry of the moduli stack $\mathcal{M}$ controls the structure of stable homotopy.

Algebra	Topology
Maps $\operatorname{Spec} R \to \operatorname{Spec} L$	$\mathbb{C}$-oriented cohomology theories
map $\operatorname{Spec} R \to \mathcal{M}$	$\mathbb{C}$-orientable cohomology theories
quasi-coherent sheaves on $\mathcal{M}$	all cohomology theories
...	...

Filling in this part of the above table is roughly what the first third of the course will cover. Before we finish, let's mention a few organizational things. There will be posted lecture notes, so no need to take them.

Post Lecture Questions

Question 2.9. Is the stack $\mathcal{M}$ separated?

A: No. In characteristic zero it looks like $B\mathbb{G}_m$. It has an affine diagonal.

Digression 2.10. A topological space is Hausdorff if and only if the diagonal embedding $X \to X \times X$ is a closed map. Being separated is a scheme-theoretic analog of this, which is appropriate even though the underlying space of a scheme is almost never Hausdorff. A scheme X (over S) is *separated* if the diagonal map $\Delta : X \to X \times_S X$ is a closed immersion. Here $X \times_S X$ is the scheme theoretic fiber product.

A representable morphism $\mathcal{F} \to \mathcal{G}$ of algebraic stacks (over S) is *universally closed* if $\mathcal{F} \times_{\mathcal{G}} Y \to Y$ is a closed immersion for all schemes Y (over S). Finally an algebraic stack is *separated* if the (representable) diagonal map $\Delta_{\mathcal{F}} : \mathcal{F} \to \mathcal{F} \times_S \mathcal{F}$ is universally closed.

In particular suppose a stack $\mathcal{F}$ has a presentation by the groupoid $\Gamma \rightrightarrows U$. Suppose further that $\mathcal{F}$ is separated. Then $\mathcal{F} \to \mathcal{F} \times_S \mathcal{F}$ is universally closed. Taking $Y = U \times_s U$, we see that

$$\Gamma = (U \times_S U) \times_S \mathcal{F} \to U \times_S U$$

must be a closed immersion. This always fails for the stack $BG = [G \rightrightarrows pt]$ (if $G \neq 1$). ⋄

3 Lazard's Theorem

Today and tomorrow will be the purely algebraic portion of the course. Recall the definition of formal group law $f(x,y) \in R[[x,y]]$ from the previous lecture (here R is a commutative ring). Given R, the collection of such formal group laws is a set $FGL(R)$. In fact this is a functor. By substituting coefficients under $R \to R'$ we get a map $FGL(R) \to FGL(R')$.

This 'group law' is supposed to correspond to a group. What is the group? The group is $Spf\mathbb{R}[[x]]$ the "formal group scheme". This is the 'space' of very small values of x. f should give a group law on the set of values x and y can take, but since f is a power series, the values must be 'very small'. This is of course heuristic. It is only a slight exaggeration to say that the precise definition is just a rewording of our previous definition. Nevertheless it is a useful perspective.

A general formal group law has the following form,

$$f(x,y) = \sum a_{i,j} x^i y^j$$

with $a_{i,j} \in R$. To be a formal group law, these coefficients must satisfy certain conditions. For example $a_{i,0} = a_{0,i} = 1$ if $i = 1$ and is zero otherwise. Also $a_{i,j} = a_{j,i}$. The associativiy also says something, but it is some (complicated) polynomial equations in the $a_{i,j}$'s. So a FGL is the same as a collection of $a_{i,j}$'s which satisfy these conditions. This isn't such a good description since we haven't actually specified what these conditions are, but it is logically equivalent.

Yet another way to say what a FGL is is that it is a map from $\mathbb{Z}[a_{i,j}]/Q$ to R, where Q is the ideal of these (still unspecified) relations.

Definition 3.1. The *Lazard ring* is $L = \mathbb{Z}[a_{i,j}]/Q$. ◇

By abuse of notation we will also let $a_{i,j}$ denote it's image in L. Then $f(x,y) = \sum a_{i,j} x^i y^j$ is a FGL for L which is universal, in the sense that

$$\mathrm{Hom}(L,R) \to FGL(R)$$

sending a homomorphism to the pushforward of $f(x,y)$ is an isomorphism.

Claim 3.2. *L is a graded ring, where $a_{i,j}$ has degree $2(i+j-1)$. This just means that if $|x| = |y| = -2$, then $f(x,y)$ also has degree -2.* ◇

Why -2? This comes from compatibility with topology. Recall that if E is a nice (i.e. complex oriented) cohomology theory, then we get a formal group law $f(x,y)$ which lives in $(\pi_{even}E)[[x,y]]$. [[★★★ This explains why $f(x,y)$ should be evenly graded, and I see that we have $\pi_2 E \cong E^{-2}(pt)$, but I still don't see why $f(x,y)$ should have degree negative two when x and y do. It seems more natural to give it degree $+2$.]]

So this claim comes down to saying that the relations respect this grading. The most complicated relations come from the associativity conditions. Note it is easy to see that L is non-negatively graded, and that $L_0 \cong \mathbb{Z}$.

There is a highbrow explanation of this. The functor FGL has an action of $\mathbb{G}_m$, i.e. for any R, we have an action of $\mathbb{G}_m(R) = \mathbb{R}^\times$ on $FGL(R)$, by $f^\lambda(x,y) = \lambda^{-1} f(\lambda x, \lambda y)$. In fact, this actually makes sense even when λ is not invertible (you have to expand the expression to see this). This means that the action extendeds to an action of all elements, not just the invertible ones. This is why the grading is non-negative. Furthermore we have $f^0(x,y) = x+y$. This reflects the fact that $L_0 \cong \mathbb{Z}$.

Theorem 3.3 (Lazard)**.** *The ring L is isomorphic to a polynomial ring $\mathbb{Z}[t_1, t_2, \dots]$ with $|t_i| = 2i$.*

Corollary 3.4. *It's easy to give formal group laws. We just need a sequence of elements in R.*

How are we going to prove this? Well the corollary says that there should be lots of formal group laws. Our first goal will be to realize this by producing lots of examples of formal group laws. Here is a recipe for how to get (some) formal group laws.

1. First look a the additive formal group law $f(x,y) = x+y$, which is a FGL over any ring.
2. Next, note that f is any FGL over R, and $g(x) = x + b_1x^2 + b_2x^3 + \cdots$, then we get a new FGL by conjugating by g. i.e. $g(f(g^{-1}x, g^{-1}y))$ is a FGL. In fact, g could start by λx for a unit λ.

This has a direct consequence when we look at the universal case. Let $R = \mathbb{Z}[b_1, b_2, \cdots]$ and let $g(x) = x + b_1x^2 + b_2x^3 + \cdots$. Then $g(g^{-1}(x) + g^{-1}(y))$ is a FGL over R. This is classified by a map $\theta : L \to R$, from the Lazard ring. This map is not an isomorphism. If it were, it would say that every formal group law comes from the additive FGL via a change of variables. However it is an isomorphism over $\mathbb{Q}$, which we will soon see.

Note: θ is a map of *graded* rings, where $|b_i| = 2i$. If x has degree -2, then it is easy to see that $g(x)$ has degree -2, and so the FGL given by g is also of degree -2. There is a similar highbrow way to see this. [[★★★ check that]]

It is helpful to linearize the problem. We have a map of schemes $\operatorname{Spec} R \to \operatorname{Spec} L$. The left side is an infinite dimensional affine space (the right side is also a infinite dimensional affine space, but we don't know that yet). There are canonical points on both sides. e.g. the canonical point $pt \to \operatorname{Spec} R$ (given by killing off all b_i) goes to the canonical point of L (given by killing off all $a_{i,j}$). By 'linerizing' we mean that we will consider the derivative of θ at these points.

Let I be the ideal in L given by all positiive degree elements. Similarly let J be the ideal of R given by all positiive degree elements. $\theta : L \to R$ maps I to J, so we get an induced map,

$$I/I^2 \to J/J^2$$

which is a map of graded abelian groups. This later is a free group on the "b_i's". So we have $\psi_n : (I/I^2)_{2n} \to (J/J^2)_{2n} \cong \mathbb{Z}$.

Lemma 3.5. *For all n the map ψ_n is injective. It's image is $\mathbb{Z}$ if $n+1 \neq p^k$ and is $p\mathbb{Z}$ if $n+1 = p^k$.*

We will come back to the proof of this lemma later. For now let's assume it.

Corollary 3.6. *For all $n > 0$, there is a (canonical) isomorphism $(I/I^2)_{2n} \cong \mathbb{Z}$.*

Since $I_{2n} \to (I/I^2)_{2n}$ is surjective we can lift the generator to an element $t_n \in L_{2n}$, and this gives a map of rings,

$$\phi : \mathbb{Z}[t_1, t_2, \ldots] \to L$$

Define the composite map:

$$\eta : \mathbb{Z}[t_1, t_2, \ldots] \xrightarrow{\phi} L \xrightarrow{\theta} \mathbb{Z}[b_1, b_2, \cdots].$$

Claim 3.7. ϕ is surjective. ◇

Proof. We show by induction on n that ϕ is a surjection in degree $2n$. By hypothesis, we know that the image of ϕ contains $(I^2)_{2n}$, and a generator for $(I/I^2)_{2n} \cong \mathbb{Z}$. The result follows. □

Lemma 3.8. *The composite map $\eta = \theta \circ \phi$ is injective. In particular ϕ is injective.*

Proof. Consider the following square.

$$\begin{array}{ccccc} \mathbb{Q}[t_1, t_2, \ldots] & & \xrightarrow{\quad \eta \otimes \mathbb{Q} \quad} & & \mathbb{Q}[b_1, b_2, \ldots] \\ \uparrow & & & & \uparrow \\ \mathbb{Z}[t_1, t_2, \ldots] & \xrightarrow{\phi} & L & \xrightarrow{\theta} & \mathbb{Z}[b_1, b_2, \ldots] \end{array}$$

Since $\mathbb{Z}[t_1, \ldots]$ and $\mathbb{Z}[b_1, \ldots]$ are torsion free, we know that they inject into their rationalizations. Notice that if the top map $\eta \otimes \mathbb{Q}$ is injective, then the bottom map must also be injective. In fact $\eta \otimes \mathbb{Q}$ is an isomorphism. We know know that $\mathbb{Q}[t_1, t_2, \ldots]$ and $\mathbb{Q}[b_1, b_2, \ldots]$ are graded vector spaces with the same dimension in each degree, and by the previous lemma $\eta \otimes \mathbb{Q}$ is surjective, hence it is an isomorphism. □

Corollary 3.9. *Lazard's theorem.*

Now we will prove Lemma 3.5, namely the map $\theta : L \to \mathbb{Z}[b_1, b_2, \cdots]$ classifying the FGL $g(g^{-1}(x) + g^{-1}(y))$ with $g = x + b_1 x^2 + \cdots$. Recall that $I \subset L$ and $J \subset \mathbb{Z}[b_1, b_2, \cdots]$ are the ideals of positively graded elements. We saw that it is sufficient to show that the induced map

$$\psi_n : (I/I^2)_{2n} \to (J/J^2)_{2n} \cong \mathbb{Z}$$

is injective. As we will see its image is $\mathbb{Z}$ if $n + 1 \neq p^k$ for a prime p and is $p\mathbb{Z}$ if $n + 1 = p^k$.

Remark 3.10. If R is a graded ring, then $\mathrm{Hom}(L, R) \cong FGL(R)$. Inside this there are the homomorphisms which preserve the grading. This corresponds to the subset of FGLs such that $f(x, y)$ has degree -2 if x and y have degree -2. We will denote this by $FGL^{gr}(R)$. ◇

Now we want to understand $(I/I^2)_{2n}$. What is $\mathrm{Hom}((I/I^2)_{2n}, M)$ where M is any abelian group?

We construct a commutative ring $\mathbb{Z} \oplus M$ with $(a, m)(a'm') = (aa', am' + a'm)$. We also give it a grading with $\mathbb{Z}$ in degree zero and M in degree $2n$. Then we see that

$$\mathrm{Hom}((I/I^2)_{2n}, M) \cong \mathrm{Hom}^{gr}(L, \mathbb{Z} \oplus M) \cong FGL^{gr}(\mathbb{Z} \oplus M) =: F(M)$$

(this defines the functor $F(M)$). By Yonneda's lemma understanding $(I/I^2)_{2n}$ is equivalent to understanding the functor F.

What are the graded FGL over $\mathbb{Z} \oplus M$? They must look like $x + y + \sum_{i+j=n+1} m_{i,j} x^i y^j$ satisfying the conditions of a FGL. Now, because there are some many fewer terms, associativity is not going to be too complicated to write down explicitly. We look at the associativity expressions in the component $x^i y^j z^k$ with all $i, j, k > 0$. On the one hand we get an expression like

$$\begin{aligned} f(f(x,y),z) &= f(x + y + \sum m_{i,j} x^i y^j, z) \\ &= x + y + z + \sum m_{i,j} x^i y^j + \sum m_{i+j,k} \left(x + y + \sum m_{i,j} x^i y^j\right)^{i+j} z^k. \end{aligned}$$

The square zero multiplication allows us to reduce the $x^i y^j z^k$ coefficient to $\binom{i+j}{j} m_{i+j,k}$. A similar expression holds for the other possible bracketing, and so we've shown the following lemma.

Lemma 3.11. *$F(M) \cong \{m_{i,j} \in M \mid i+j = n+1$ and conditions (1), (2), and (3) below hold$\}$.*

1. $m_{0,n+1} = m_{n+1,0} = 0$,
2. $m_{i,j} = m_{j,i}$,
3. $\binom{j+k}{j} m_{i,j+k} = \binom{i+j}{j} m_{i+j,k}$.

What are the solutions to these equations? The homomorphism $L \to \mathbb{Z}[b_1, b_2, \cdots]$ induces a map $\psi_n : (I/I^2)_{2n} \to (J/J^2)_{2n}$ and so we get a map

$$M \cong \mathrm{Hom}((J/J^2)_{2n}, M) \to F(M)$$

This gives us a collection of 'obvious' elements of $F(M)$, and hence 'obvious' formal group laws on $\mathbb{Z} \oplus M$. An element $m \in M$ on the left-hand side corresponds to a graded ring homomorphism from $\mathbb{Z}[b_1, b_2, \cdots]$ to $\mathbb{Z} \oplus M$, and the polynomial g is sent under this map to $g(x) = x + mx^{n+1}$. Since $m^2 = 0$ in $\mathbb{Z} \oplus M$, we have $g^{-1}(x) = x - mx^{n+1}$. This allows us to compute the formal group laws (and hence obvious solutions) as

$$f(x,y) = x + y - m(x^{n+1} + y^{n+1}) + m(x+y)^{n+1}.$$

So the obvious solutuion for an element $m \in M$ has $m_{i,j} = \binom{n+1}{i} m$ if $i, j \neq 0$ and zero if either $i = 0$ or $j = 0$.

'Non-obvious' solututions also occur when we divide by $d = \gcd\{\binom{n+1)}{i}\}_{0<i<n+1}$. We therefore would like to understand what these common denominators are. We have the following combinatorial lemma.

Lemma 3.12. *Let p be a prime number, and let $a = \sum a_i p^i$ and $b = \sum b_i p^i$ be the p-base expansions of a and b. Then $\binom{a}{b}$ is congruent to $\prod \binom{a_i}{b_i}$ (mod p).*

Proof. Let S be a set of size a and partition it into subsets $S = \sqcup S_\alpha$ where exactly a_i subsets have size p^i. There is an action on each S_α be a group $G_\alpha = \mathbb{Z}/p^i\mathbb{Z}$ (for appropriate i). Thus the product $G = \prod G_\alpha$ acts on S.

Let T be the collection of subsets of S of size b. We have $|T| = \binom{a}{b}$. The set T is also acted upon by G (the action is induced from the one on S). Moreover G is a p-group so every (non-trivial) orbit has size divisible by p. Thus $|T| \equiv |T^G| \mod p$, where T^G is the set of fixed points under the G-action.

A fixed point in T is precisely a subset $S_0 \subset S$ with exactly b elements, which is a union of some of the subsets S_α. There are precisely $\prod \binom{a_i}{b_i}$ ways to choose these subsets. □

Corollary 3.13. *$\binom{i+j}{i} \equiv 0$ mod p is equivalent to the statement that some p-digit of i is larger then the corresponding p-digit in $i+j$. This is in turn equivalent to the statement that when the sum $i+j$ is computed in base p, some "carrying" must occur.*

Corollary 3.14. *$d = \gcd\{\binom{n+1}{i}\}_{0<i<n+1}$ is either a prime p (if $n+1 = p^k$) or is 1 (otherwise).*

Proof. If $n+1$ is not a power of a prime p, then we can decompose it as $i+j$ in a non-trivial way such that the sum can be computed without carrying. Hence $\binom{n+1}{i}$ is not divisible by p. If $n+1 = p^k$, then there is no such decomposition, so p is a common divisor of the set $\{\binom{n+1}{i}\}_{0<i<n+1}$. Exercise B.1 then shows that p must be the *greatest* common divisor. □

Let $\lambda : M \to F(M)$ be the map which corresponds to the obvious solutions. The above shows that we get a factorization of λ as,

$$M \xrightarrow{\cdot d} M \xrightarrow{\lambda'} F(M),$$

where λ' consists of the 'non-obvious' solutions. The following proposition then proves Lemma 3.5.

Proposition 3.15. *The map $\lambda' : M \to F(M)$ (consisting of the non-obvious solutions) is an isomorphism.*

Proof. To prove this, it suffices to prove it locally, i.e. localizing at each prime separately. We want to prove that λ' induces an isomorphism

$$M_{(p)} \to F(M)_{(p)} \cong F(M_{(p)}.$$

Thus we may assume that M is a $\mathbb{Z}_{(p)}$-module. In this case the collection of $\{m_{i,j}\} \in F(M)$ simplifies. We get the following lemma.

Lemma 3.16. *Let $\{m_{i,j}\} \in F(M)$. then*

1. *If $m_{i,j} = 0$, then $m_{j,i} = 0$.*
2. *if $i, k > 0$ and $i+k$ is computed in base p without carrying, then $m_{i,j} = 0$ implies $m_{i+k,j-k} = 0$.*

Proof. Part (1) is trivial. Part (2) follows, because we have

$$\binom{i+k}{k} m_{i+k,j-k} = \binom{j}{j-k} m_{i,j} = \binom{j}{k} m_{i,j} = 0.$$

Thus since $\binom{i+k}{k}$ is not divisible by p (by Corollary 3.13) we have $m_{i+k,j-k} = 0$. □

We now continue with the proof of Proposition 3.15. Consider first the prime power case where $1 + n = p^k$. Define $\chi : F(M) \to M$ sending the collection $\{m_{i,j}\}$ to the specific coefficient $m_{p^{k-1},p^k-p^{k-1}}$. Because

$$\frac{\binom{p^k}{p^{k-1}}}{p} = \frac{1}{p} \cdot \frac{p \cdot (p-1) \cdot (p-2) \cdot \cdots \cdot (p^{k-1}+1) \cdot (p^{k-1}) \cdot (p^{k-1}-1) \cdots}{p^{k-1} \cdot (p^{k-1}-1) \cdots}$$

has no p factor, it is a unit in $\mathbb{Z}_{(p)}$. Hence then the composition $\chi \circ \lambda' : M \to F(M) \to M$ is an isomorphism.

It is then enough to show that the kernel of χ is zero, i.e. if $\{m_{i,j}\} \in F(M)$ and $m_{p^{k-1},p^k-p^{k-1}} = 0$, then the whole sequence is zero. Consider a coefficient $m_{i,j}$ (where $i+j = n+1 = p^k$). By symmetry it is enough to consider just those i for which $i \geq \frac{1}{2}p^k \geq p^{k-1}$, i.e. we must show that $m_{p^{k-1}+\ell,p^k-p^{k-1}-\ell} = 0$ for $p^k - p^{k-1} \leq \ell \leq p^k$. Note that for these values of ℓ, the sum $p^{k-1}+\ell$ may be computed in base p without carrying, hence by Lemma 3.16 each of these coefficients vanishes.

Now consider the case $n + 1 \neq p^k$. Let p^e be the largest power of p dividing $n + 1$. We essentially repeat the same process we used above. Consider the map $\chi : F(M) \to M$ given by looking at the term $m_{p^e,\ell}$ (where $\ell = n + 1 - p^e$). The induced map $\chi \circ \lambda' : M \to M$ is multiplication by $\binom{n+1}{p^e}\frac{1}{d}$. In this case d is either 1 or a prime other than p. Corollary 3.13 ensures that there is no p factor in this product, so again $\chi \circ \lambda$ is an isomorphism. Hence λ' is injective, and we need to know that χ is also injective.

Now suppose that we have a sequence $\{m_{i,j}\} \in F(M)$, such that $m_{p^e,\ell} = 0$. We want to show that all $m_{i,j} = 0$. If $e > 0$, then by symmetry we have $m_{\ell,p^e} = 0$. Moreover we have

$$n + 1 - p^{e-1} = n + 1 - p^e + p^e - p^{e-1} = (n + 1 - p^e) + (p - 1)p^{e-1}.$$

This sum can be computed in base p without carrying, hence $m_{n+1-p^{e-1},p^{e-1}} = 0$.

More generally (or if $e = 0$), choose a non-trivial decomposition $n + 1 = i + j$. We wish to show that $m_{i,j} = m_{j,i} = 0$. Since $n + 1$ has a non-zero p^e-digit, then either i or j (hence without loss of generality i) has a non-zero p^e-digit or a non-zero p^{e-1}-digit. Let $a = e$ or $e - 1$ depending on the respective case. We have,

$$i = (i - p^a) + p^a$$

is a sum which does not involve carrying in base p. Since we already know that $m_{p^a,n+1-p^a} = 0$, we conclude from Lemma 3.16 that $m_{i,j} = 0$. □

4 Complex Oriented Cohomologies and Spectra

We are going to be assuming basic knowledge of spectra and cohomology theories. The words spectrum and cohomology theory will more or less be used interchangeably. Given a spectrum $\mathbb{E}$, we get a cohomology theory via $E^n(X) = [X, \Omega^{\infty-n}E] \cong \pi_{-n}E^X$. There is also a homology theory $E_n(X) = \pi_n(E \otimes (\Sigma^\infty_+(X))$. These groups can also be defined when X is a spectrum.

A multiplicative cohomology theory is a ring spectrum $E \otimes E \to E$ with unit $S \to E$ i.e. $1 \in \pi_0(E)$. These will be associative up to homotopy. Most will also be commutative up to homotopy. Some will *not* be structured ring spectra so it is inconveinient to build that structure into the framework. In the overview we saw the definition of a complex orientable cohomology theory. Now we are going to introduce an alternative definition.

Definition 4.1. E is *complex orientable* if $E^*(\mathbb{CP}^\infty) \to E^*(S^2)$ is surjective. ◇

Remark 4.2. If E is complex orientable in the sense of Definition 2.2, then it is complex orientable in the above sense. This follows from the naturality and multiplicativity of the Atiyah-Hirzebruch spectral sequence. Conversely, as we shall see later, for multiplicative cohomology theories the above condition implies the degeneration of the AHSS. ◇

For pointed spaces X we have canonical splittings $E^*(X) \cong E^*(pt) \oplus \tilde{E}^*(X)$. The sphere and $\mathbb{CP}^\infty$ are pointed spaces and so the surjectivity of the above map is the same as surjectivity of this map in (reduced) $\tilde{E}^*$-cohomology. Reduced cohomology is easier since $\tilde{E}^*(S^2) \cong E^{*-2}(pt)$. This is a free $E^*(pt)$-module in degree 2. Hence this condition is equivalent to:

Definition 4.3. E is complex orientable if the canonical generator $\bar{t} \in \tilde{E}^2(S^2)$ can be lifted to a class $t \in \tilde{E}^2(\mathbb{CP}^\infty) \subset E^2(\mathbb{CP}^\infty)$. In this case such a lift is a *complex orientation.* ◇

Example 4.4. Since the canonical map is an isomorphism,

$$H^2(\mathbb{CP}^\infty) \cong H^2(S^2)$$

ordinary cohomology is (uniquely) complex orientable. ◇

More generally, we can think of complex orientations in terms of obstruction theory. A complex orientation is the same as a solution to the following lifting problem.

$$\begin{array}{ccc} (S^2, pt) & \longrightarrow & \Omega^{\infty-2}E \\ \big\downarrow & \nearrow & \\ (\mathbb{CP}^\infty, pt) & & \end{array}$$

We can use the filtration $S^2 \subset \mathbb{CP}^2 \subset \cdots \subset \mathbb{CP}^\infty$ to do obstruction theory. At each stage of the obstruction theory process there is a single obstruction which lives in $\pi_{2n+1}\Omega^{\infty-2}E = \pi_{2n+3}E = E^{-2n-3}(pt)$. If the odd homotopy groups of $\Omega^{\infty-2}E$ vanish, then there is no obstruction.

Example 4.5. Complex K-theory is complex orientable. ◇

Postnikov Sections and Truncations

Every spectrum has truncations $E \to \tau_{\leq 0}E$ whose fiber is the connective cover of E and which kills off the positive homotopy groups. There is also the dual construction $\tau_{\geq 0}E \to E$. This generalizes to all n and we have a cofiber/fiber sequence

$$\tau_{\geq n}E \to E \to \tau_{\leq n-1}E.$$

These constructions are functorial and $\tau_{\geq n}$ is the right-adjoint to the inclusion from $(n-1)$-connected spectra to all spectra. $\tau_{\leq n-1}$ is the left-adjoint to the same inclusion.

If E is a ring spectrum, then $\tau_{\geq 0}E$ is again ring spectrum. The product map is given by the following diagram,

$$\begin{array}{ccc} \tau_{\geq 0}E \otimes \tau_{\geq 0}E & \dashrightarrow & \tau_{\geq 0}E \\ \downarrow & & \downarrow \\ E \otimes E & \longrightarrow & E \end{array}.$$

The above lift exists (up to homotopy) by the universal property of the truncation. The sphere spectrum is already 0-truncated $\mathbb{S} = \tau_{\geq 0}\mathbb{S}$, and so we have a factorization of the unit map, $\mathbb{S} \to \tau_{\geq 0}E \to E$. The first map gives the unit of $\tau_{\geq 0}E$.

Exercise 4.1.[Exercise B.2] Moreover if E is compex orientable, then so is $\tau_{\geq 0}E$.

Degeneration of the Atiyah-Hirzebruch Spectral Sequence

Let X ba a space and E a cohomolgy theory. Recall the Atiyah-Hirzebruch spectral sequence (see Appendix A) with E^2-term $H_p(X; \pi_q E) \Rightarrow E_{p+q}(E)$. If E is a multiplicative cohomology theory, then the unit map $\mathbb{S} \to E$ induces a map $H\mathbb{Z} = \tau_{\leq 0}\mathbb{S} \to \tau_{\leq 0}E$. Thus ordinary integral cohomology classes induce classes in $(\tau_{\leq 0}E)$-cohomology.

Proposition 4.6. *Let X be a space and E a multiplicative cohomology theory. Assume that $H_*(X;\mathbb{Z})$ is free in each degree n on gnerators $\{h_\alpha\}_{\alpha \in B_n}$. Let $\{c_\alpha\}$ in $H^n(X;\mathbb{Z})$ be the dual elements defined by $c_\alpha(h_\beta) = \delta_{\alpha\beta}$. Let $h'_\alpha \in (\tau_{\leq 0}E)_n(X)$ and $c'_\alpha \in (\tau_{\leq 0}E)^n(X)$ be the images under the map $H\mathbb{Z} \to \tau_{\leq 0}E$. Now consider the following conditions,*

(a) each of the classes h'_α lifts to a class $h''_\alpha \in E_n(X)$,

(b) $H_*(X,\mathbb{Z})$ is finitely generated in each degree (so that the c_α form a basis) and each of the classes c'_α lifts to a class $c''_\alpha \in E^n(X)$.

(c) each of the classes c'_α lifts to a class $c''_\alpha \in E^n(X)$

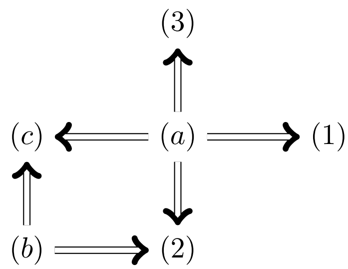
Moreover consider the following statements,

1. *$E \otimes (\Sigma^\infty_+ X) \simeq \oplus_{\alpha \in B_n} \Sigma^n E$ as E-modules,*
2. *$E^X \simeq \prod_{\alpha \in B_n} \Sigma^{-n} E$*

3. There are non-canonical isomorphisms

$$E_*(X) \simeq H_*(X; \mathbb{Z}) \otimes (\pi_* E)$$
$$E^*(X) \simeq \mathrm{Hom}(H_*(X; \mathbb{Z}), \pi_* E)$$

Then we have the following diagram of implications:



Proof. Condition (c) is clearly a consequence of condition (b), and statement (3) follows from the combination of statements (1) and (2). We will first prove that (a) implies statement (1). The class $h''_\alpha \in E_n(X)$ is represented by a homotopy class of maps of spectra $h''_\alpha : S^n = \Sigma^n \mathbb{S} \to E \otimes (\Sigma^\infty_+ X)$. Tensoring by the identity yields $E \otimes S^n \to E \otimes E \otimes (\Sigma^\infty_+ X)$, and using the multiplication $E \otimes E \to E$ we get a composite map, $\Sigma^n E \to E \otimes (\Sigma^\infty_+ X)$. The coproduct of these maps yields a comparison map

$$E \otimes (\Sigma^\infty(\vee_{\alpha \in B_n} S^n)) \cong \oplus_{\alpha \in B_n} \Sigma^n E \to E \otimes (\Sigma^\infty_+ X), \tag{4.7}$$

which is an E-module map. What remains is to show that this map is an equivalence of spectra. It is enough to prove 4.7 is an isomorphism on homotopy groups.

We will prove this by comparing these spectra via the Atiyah-Hirzebruch spectral sequence. The homological AHSS for the space X converges to $\pi_*(E \otimes X) = E_*(X)$. It arrises (if X is a CW-complex) from the skeletal filtration of X,

$$X_0 \subset X_1 \subset X_2 \subset \cdots \subset X.$$

To compare the spectra in Equation (4.7), we need to compare the skeletal filtrations of X and $(\vee_{\alpha \in B_n} S^n)$. There is generally no map between the individual filtration spaces, nor a map $(\vee_{\alpha \in B_n} S^n) \to X$ inducing the above map of spectra. However, each of the skeleta X_n satisfies the conditions of (a). Thus for each $k \geq 0$ we have a map of spectra,

$$E \otimes (\Sigma^\infty(\vee_{\alpha \in B_n} S^n)_{n \leq k}) \to E \otimes (\Sigma^\infty_+ X_k).$$

These maps are compatible and show that the map (4.7) is actually a map of filtered spectra. Hence we get a map of the induced spectral sequences (i.e. the AHSSs). The conditions on X (namely (a)) show this is an isomorphism on E^2-terms, hence E^∞-terms, hence (4.7) induces an isomorphism of homotopy groups. [[★★★ See Jacob's notes for an essentially equivalent proof.]]

What we have learned is that if we can construct a comparison maps of the relevant spectra in sufficient generality/functoriality, then the mere existence of the Atiyah-Hirzerbuch spectral sequence forces these comparison maps to be equivalences. This holds in the dual situation as well, as we shall now see that (b) implies (2).

The classes $c''_\alpha \in E^n(X)$ represent homotopy classes of maps of spectra $\Sigma^{-n}\mathbb{S} \to E^X$. By adjunction, this is the same as a map $\Sigma^{\infty-n}_+ X \to E$. This yields, using the multiplication in E, an E-module map $\Sigma^{-n}E \otimes (\Sigma^\infty_+ X) \to E$, or by adjunction $\Sigma^{-n}E \to E^X$. Since there are finitely many α in each degree taking the coproduct[1] yields a comparison map,

$$E^{\vee_{\alpha \in B_n} S^n} \cong \prod_{\alpha \in B_n} \Sigma^{-n}E \to E^X.$$

An essentially identical argument using the cohomological AHSS shows that this is an isomorphism on all homotopy groups, and hence an equivalence of spectra.

Alternatively, if condition (a) is satisfied. Then condition (c) is satisfied as well as statement (2)[2] Although many of the multiplicative cohomology theories we will encounter will come from structured ring spectra, we will avoid using this in the following argument. Given a homotopy associative and unital ring spectrum E, and two homotopy associative E-module spectra N and M, the set $[M, N]_E \subseteq [M, N]$ consists of those homotopy classes of maps of spectra which are compatible with the E-module structure in the homotopy category. The E-module spectrum $E \otimes (\Sigma^\infty_+ X)$ has the property that $[E \otimes (\Sigma^\infty_+ X), N]_E = [(\Sigma^\infty_+ X), N]$ for all E-module spectra N.

We have seen that conditions (a) induce an equivalence of E-module spectra $E \otimes X = \oplus \Sigma^n E$. Thus, for any spectrum Y we have natural bijections,

$$\begin{aligned}
[Y, E^X] &= [(\Sigma^\infty_+ X) \otimes Y, E] \\
&= [E \otimes (\Sigma^\infty_+ X) \otimes Y, E]_E \\
&\cong [(\oplus \Sigma^n E) \otimes Y, E]_E \\
&\cong \prod [E \otimes \Sigma^n Y, E]_E \\
&\cong \prod [\Sigma^n Y, E] \\
&\cong \prod [Y, \Sigma^{-n} E] \\
&\cong [Y, \prod \Sigma^{-n} E]
\end{aligned}$$

Thus by Yoneda, there exists a equivalence of spectra $E^X \simeq \prod_{\alpha \in B_n} \Sigma^{-n}E$. The de suspensions of the unit maps from the left hand side give homotopy classes of maps $c''_\alpha : \Sigma^{-n}\mathbb{S} \to \prod_{\alpha \in B_n} \Sigma^{-n}E \simeq E^X$, which are the same as equivalence to classes,

$$c''_\alpha \in E^n(X) = [X, \Sigma^n E] = [\Sigma^{-n}\mathbb{S}, E^X].$$

Thus we see that (a) implies both (c) and (2). [[★★★ In lecture it was claimed that (b) implies (1), but I don't yet understand that implication. Adam's book [Ada95] says this can be proved using the duality pairing between the homology and cohomology AHSSs.]] □

Remark 4.8. As a corollary of the above proof we see that under these conditions the Atiyah-Hirzebruch spectral sequence collapses for X. ⋄

[1] The canonical map from a finite coproduct to the corresponding product is an equivalence of spectra.

[2] The following argument was kindly supplied by Tyler Lawson via a personal communication.

Assume that E is complex oriented, so that we have a class $t \in E^2(\mathbb{CP}^\infty)$. Recall that the homology of a finite projective space is $H^*(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}[t]/(t^{n+1})$. This t lifts to E-cohomology (since E is complex oriented and we have a factorization $S^2 \to \mathbb{CP}^n \to \mathbb{CP}^\infty$), hence the classes $1, t, t^2, \dots$ also lift to E-cohomology. This means that $\mathbb{CP}^n$ satisfies the conditions of the proposition, so that $E^*(\mathbb{CP}^n)$ is a free $\pi_*(E)$-module on the basis $1, t, t^2, \dots$. Even better, we have the following lemma.

Lemma 4.9. *We have an isomorphism $E^*(\mathbb{CP}^\infty) \cong (\pi_* E)[t]/(t^{n+1})$. i.e. $t^{n+1} = 0$ holds in $E^{2n+2}(\mathbb{CP}^n)$.*

Proof. Without loss of generality we can replace E by $\tau_{\geq 0}E$, i.e. assume E is connective. This is because $\tau_{\geq 0}$ is a right adjoint. Then $\Omega^{\infty-2n-2}E$ is $(2n+1)$-connective. Hence $t^{n+1} : \mathbb{CP}^n \to \Omega^{\infty-2n-2}E$ is null-homotopic. □

This means that the E-cohomology of $\mathbb{CP}^\infty$ can be computed as

$$E^*(\mathbb{CP}^\infty) = E^*(\operatorname{colim} \mathbb{CP}^n) \to \lim E^*(\mathbb{CP}^n) \cong (\pi_* E)[[t]].$$

In general, the cohomology of the colimit maps to the to the limit of the cohomologies, but there will be a long exact sequence with $\lim^1$ terms. These terms vanishes in this case, since all the maps in the limit are surjective. Hence this is an isomorphism.

Remark 4.10. This shows that for a complex oriented cohomology theory in the sense of Definition 4.1, the Atiyah-Hirzebruch spectral sequence collapses, and hence, by our previous considerations, this is equivalent to Definition 2.2. ◇

Note that $\mathbb{CP}^\infty$ can be realized as a topological monoid (it is a group, but inverse is not continuous). For example consider the vector space $V = \mathbb{C}(x)$, which is a topological monoid by multiplication of polynomials. We can realize $\mathbb{CP}^\infty$ as $\mathbb{CP}^\infty = V \smallsetminus \{0\}/\mathbb{C}^\times$. This induces a multiplication on $\mathbb{CP}^\infty$. Thus we get a map,

$$\begin{aligned} m^* &: E^*(\mathbb{CP}^\infty) \to E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \\ m^* &: (\pi_* E)[[t]] \to (\pi_* E)[[u, v]] \\ & t \mapsto f(u, v) \end{aligned}$$

The conclusion is that E together with the class $t \in E^2(\mathbb{CP}^\infty)$ determines a graded commutative ring $R = \pi_*(E)$ together with a graded formal group law $f(u, v) \in R[[u, v]]$.

Given a complex oriented cohomology theory E, we can also compute the E-cohomology of many spaces related to $\mathbb{CP}^\infty$. Consider the spaces $BU(n)$. There is a map

$$\Theta : BU(1)^n \to BU(n),$$

which classifies taking n complex line bundles and forming the direct sum. This is Σ_n-equivariant up to homotopy. So we get a map

$$H^*(BU(n), \mathbb{Z}) \to \left(H^*(\mathbb{CP}^\infty; \mathbb{Z})^{\otimes n}\right)^{\Sigma_n} \cong (\mathbb{Z}[t_1, \dots, t_n])^{\Sigma_n} \cong \mathbb{Z}[c_1, \dots, c_n]$$

where the c_i are the n^{th} elementary symmetric polynomials. This map is an isomorphism in ordinary cohomology and these classes c_i are the Chern classes. We can also consider the dual situation. The homology of projective space is $H_*(\mathbb{CP}^\infty) = \mathbb{Z}\{b_0, b_1, \dots\}$, where these are the dual basis elements to the t^k. So we get a map

$$(\mathbb{Z}\{b_i\}^{\otimes n})_{\Sigma_n} \to H_*(BU(n);\mathbb{Z})$$

which is also an isomorphism. Thus $H_*(BU(n);\mathbb{Z}) \cong \mathrm{Sym}^n \mathbb{Z}\{b_0, b_1, \cdots\}$.

For a complex oriented cohomology theory, the situation is very similar. We have $E^*(\mathbb{CP}^\infty) \cong (\pi_* E)[[t]]$, and there are dual elements β_i (dual to the t^i) which form a basis for $E_*(\mathbb{CP}^\infty)$ as a $\pi_* E$-module. [[★★★ I still don't see how this last statement follows from having the complex orientation]]. As with ordinary homology Θ induces a map

$$\mathrm{Sym}^n_{\pi_*(E)} \pi_* E\{b_0, b_1, \cdots\} \to E_*(BU(n))$$

which is an isomorphism.

5 The Complex Bordism Spectrum MU

Here are some recollections about ordinary cohomology. We have the following isomorphisms.

$$\begin{aligned} H^*(\mathbb{CP}^\infty;\mathbb{Z}) &= \mathbb{Z}[[t]] = \mathbb{Z}\{t^i\} \\ H_*(\mathbb{CP}^\infty;\mathbb{Z}) &= \mathbb{Z}\{b_i\} \\ H^*(BU(n);\mathbb{Z}) &= (\mathbb{Z}[t_1, \cdots, t_n])^{\Sigma_n} = \mathbb{Z}[[c_1, \dots, c_n]] \\ H_*(BU(n);\mathbb{Z}) &= \mathrm{Sym}^n H_*((\mathbb{CP}^\infty)^n;\mathbb{Z}) \cong \mathbb{Z}\{b_{i_1}, \dots, b_{i_n}\} \end{aligned}$$

Similarly we saw last time that if E is complex oriented, then we get essentially the same answer with $\mathbb{Z}$ replaced by $\pi_* E$. Notice that there is a diagonal map $X \to X \times X$ which gives,

$$E_*(X) \to E_*(X \times X) \leftarrow E_*(X) \otimes_{\pi_* E} E_*(X)$$

If this second map, induced by the projections, is an isomorphism, then we get a comultipliation on $E_*(X)$. When does this happen? We saw that if X satisfies the conditions outlined last time, then this second map will be an isomorphism.

For complex oriented E we have $E^*(\mathbb{CP}^\infty) \cong (\pi_* E)[[t]]$ and $E_*(\mathbb{CP}^\infty) \cong (\pi_* E)[b_0, b_1, \dots]$. The coalgebra structure on E-homology is exactly like it is in ordinary cohomology

$$\Delta(b_n) = \sum_{i+j=n} b_i \otimes b_j.$$

Morevoer we have an isomorphism $E^*(BU(n)) \cong (\pi_* E)[[c_1, \dots, c_n]]$. The c_i are the E-Chern classes. They satisfy most of the usual properties of the usual ordinary Chern classes. They are dermined by the splitting principle in exactly the same way as the usual Chern classes. In particular we get

$$c_n(V \oplus W) = \sum_{i+j=n} c_i(V) c_j(W)$$

Where this theory differs is in how it behaves on tensor products of vector bundles. It can be reduced (in the same way that ordinary Chern classes can be reduced) to the tensor proodcut of line bundles. This is governed by a FGL, as we've discussed earlier.

Question 5.1. How do we see that the H-structure on $\mathbb{CP}^\infty$ given by tensor product of lines is the same as the product we saw before?

The multiplication in $\mathbb{CP}^\infty$ is induced by multiplication in the vector space $V = \mathbb{C}(t)$. The space $V \smallsetminus \{0\}$ is also the universal principal $\mathbb{C}^\times$ bundle over $\mathbb{CP}^\infty$. The multiplication $V \times V \to V$ is a bilinear map, and a quick calculation shows that the bundle $V \otimes V$ over $\mathbb{CP}^\infty \times \mathbb{CP}^\infty$ coincides with the pull-back of V via the multiplication map $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$.

Orientations

Let X be a space and V be a (real) vector bundle over X with a metric. Furthermore, let E be a multiplicative cohomology theory. The metric allows us to form the unit ball and unit sphere bundles over X, $B(V) \to X$ and $S(V) \to X$.

Definition 5.2. *Cohomology twisted by* V is defined as $E^{*-V}(X) = E^*(B(V), S(V)) \cong \tilde{E}^*(X^V)$ where $X^V = B(V)/S(V)$ is the Thom space of V. ◇

Example 5.3. If the vector bundle is the trivial bundle $\underline{\mathbb{R}}^n$ then the Thom space is $X^{\underline{\mathbb{R}}^n} \cong \Sigma^n X$ and so $E^{*-\underline{\mathbb{R}}^n}(X) = \cong E^{*-n}(X)$. ◇

This last example shows how trivial bundles correspond to the usual grading of cohomology. More generally we can grade cohomology by general vector bundles on X.

Definition 5.4. An E-orientation of V is a class $u \in E^{n-V}(X)$ such that for every $x \in X$,

$$u \mapsto u_x \in E^{n-V_x}(\{x\}) \cong E^0(pt)$$

is a unit in the ring $E^0(pt)$. (This identification depends slightly on the choice of trivialization of V_x). Such a class u, if it exits, is called a *Thom class.* ◇

It suffices to check this condition at one point in each component of X. [[★★★ Why?]] A Thom class induces isomorphisms by multiplication by u,

$$E^*(X) \to E^{*+n-V}(X) \cong \tilde{E}^{*+n}(X^V).$$

These are called *Thom isomorphisms.*

Example 5.5. Let $X = \mathbb{CP}^\infty$. Then $BV \simeq X$ (this is always true for any vector bundle and any space) and $SV \cong EU(1) \simeq pt$. Thus, $(\mathbb{CP}^\infty)^V \simeq \mathbb{CP}^\infty$ and we have,

$$E^{2-V}(\mathbb{CP}^\infty) = \tilde{E}^2(\mathbb{CP}^\infty) \to \tilde{E}^2(S^2) \cong \tilde{E}^0(S^0).$$

Thus the class $t \in \tilde{E}^2(\mathbb{CP}^\infty)$ for a complex oriented theory E is the same as an E-orientation of V on $\mathbb{CP}^\infty$. ◇

So a choice of orientation for E gives us a chioce of orientation in the above sense for the universal bundle V over $\mathbb{CP}^\infty$.

Remark 5.6. If $v \in H^{m-V}(X)$ and $w \in H^{n-W}(X)$, then $vw \in H^{m+n-(V \oplus W)}(X)$ is an orientation of $V \oplus W$. ◇

Now let E be a complex oriented cohomology theory. Then

Claim 5.7. Every complex vector bundle on every space X has a canonical Thom class. $\diamond$

Proof. Without loss of generality we can assume we are in the universal case $X = BU(n)$ and $V =$ the universal bundle. Then $BU(n) = EU(n)/U(n)$ and $BU(n-1) = EU(n)/U(n-1)$ and we get a fiber bundle

$$U(n)/U(n-1) = S^{2n-1} \to BU(n-1) \to BU(n).$$

Thus $E^{*-V}(BU(n)) \cong E^*(BU(n), BU(n-1))$ and we get

$$\begin{aligned} E^*(BU(n), BU(n-1)) \to E^*(BU(n)) \to E^*(BU(n-1)) \to \dots \\ c_n E^*(BU(n)) \to \pi_* E[[c_1, \dots, c_n]] \to \pi_* E[[c_1, \dots, c_{n-1}]] \end{aligned}$$

The long exact sequence splits into short exact sequences since $E^*(BU(n)) \to E^*(BU(n-1))$ is surjective. Thus the relative cohomology group has a canonical class c_n.

Claim 5.8. c_n is a canonical orientation of $V \to BU(n)$ $\diamond$

Proof. The map $BU(1)^n \to BU(n)$ (classifying the direct sum of lines) is surjective on π_0, so it suffices to check the orientation condition on $BU(1)^n$. Here it is also enough to check it on each line individually. The class c_n restricts to the first Chern class of each line, and so this is exactly the computation we did in the last example. $\square$

This completes the proof of Claim 5.7. $\square$

This is how complex oriented cohomology theories get their name. We have orientations for each complex vector bundle.

The Thom Spectrum MU: Complex Bordism

Given the vector bundle V over $BU(n)$, we can form the Thom space and then the suspension spectrum $MU(n) = \Sigma^{\infty-2n} BU(n)^V$. These spectra satisfy a universal property. By construction, $E^{2n-V}(BU(n) \cong [MU(n), E]$. In particular, the classes c_n give maps $\varphi_n : MU(n) \to E$.

Note that we have maps $BU(n-1) \to BU(n)$ which classify adding an extra trivial bundle. These induce maps of Thom spaces

$$\Sigma BU(n-1)^{V_{n-1}} \to BU(n)^{V_n},$$

which in turn induce maps of spectra,

$$MU(n-1) \to MU(n).$$

Claim 5.9. The diagram

$$\begin{array}{ccc} MU(n-1) & \xrightarrow{\phi_{n-1}} & E \\ \downarrow & \nearrow_{\phi_n} & \\ MU(n) & & \end{array}$$

commutes. ◇

Proof. next time. □

We'll prove this shortly. As a consequence we can define $MU = \operatorname{colim} MU(n)$ which then maps into E (There is some subtlety here, which we'll discuss soon). MU is called *complex bordism*. MU is the Thom spectrum over $BU = \operatorname{colim} BU(n)$. It is not the Thom space/spectra of any particular vector bundle by rather the virtual vector bundle which is defined on $BU(n)$ as $V - \varepsilon^n$.

The reason that MU is called complex bordism is because it also lives in geometry. From the Thom-Pontryagin construction $\pi_* MU$ is the cobordism ring for stably almost complex manifolds. The word stably refers to the fact that we can take the tangent bundle of M and add a large trivial bundle before adding a complex structure. The *almost* means we aren't requiring any integrality condition. More generally $MU_*(X)$ is the bordism group of stably almost complex manifolds with a map to X. There are identified if there is a bordism between them, with a map to X. We will never use this description of MU in this class.

If E is a complex oriented cohomology theory, then we get maps $\varphi_n : MU(n) \to E$ from the classes c_n associated to the universal complex vector bundle over $BU(n)$.

Example 5.10. Let's look at the first two spectra $MU(n)$. We have $MU(0) = \mathbb{S}$, the suspension spectrum of S^0. The classifying space $BU(1) = \mathbb{CP}^\infty$, which is also the Thom space. Hence $MU(1) = \Sigma^{\infty-2}\mathbb{CP}^\infty$. There is a map $MU(0) \to MU(1)$ given by

$$\mathbb{S} = \Sigma^{\infty-2} S^2 \to \Sigma^{\infty-2}\mathbb{CP}^\infty.$$

Recall that $\varphi_1 : MU(1) \to E$ corresponds to the orientation class $t \in E^0(MU(1)) \cong \tilde{E}^2(\mathbb{CP}^\infty)$. The composition $\mathbb{S} = MU(0) \to MU(1) \to E$ is the unit map of E. ◇

Note that for all m, n there is a map $BU(m) \times BU(n) \to BU(m+n)$ (classifying direct sum of bundles). This gives a map of spectra $MU(m) \otimes MU(n) \to MU(m+n)$. The inclusion $MU(n) \to MU(n+1)$ can be identified with the map

$$MU(n) = MU(n) \otimes \mathbb{S} = MU(n) \otimes MU(0) \to MU(n) \otimes MU(1) \to MU(n+1).$$

Using this, the maps φ_n are uniquely determined by two requirements:

1. φ_1 is the complex orientation,
2. the following diagram commutes up to homotopy,

$$\begin{array}{ccc} MU(m) \otimes MU(n) & \longrightarrow & MU(m+n) \\ \big\downarrow {\scriptstyle \varphi_m \otimes \varphi_n} & & \big\downarrow {\scriptstyle \varphi_{m+n}} \\ E \otimes E & \longrightarrow & E \end{array}.$$

To see this we recall that by the Thom isomorphism maps we have an inclusion,

$$E^*(MU(m) \otimes MU(n)) \hookrightarrow E^{*+2m+2n}(BU(m) \times BU(n)) \cong (\pi_* E)[[c_1, \ldots, c_m, c'_1, \ldots, c'_n]]$$

and the image is the ideal generated by the class $c_m c'_n$. Hence we can consider the following diagram.

$$\begin{array}{ccc} E^*(MU(m+n)) & \hookrightarrow & E^{*+2m+2n}(BU(m+n)) \\ \downarrow & & \downarrow \\ E^*(MU(m)\otimes MU(n)) & \hookrightarrow & E^{*+2m+2n}(BU(m)\times BU(n)) \end{array}$$

We want to show that the element ϕ_{m+n} in the upper-left-hand corner which maps to $\phi_m \otimes \phi_n$ in the lower-left-hand corner. By injectivity it is enough to check the image in the lower right hand corner. This is the correct element since $c_{m+n}(\zeta \oplus \zeta') = c_m(\zeta)c_n(\zeta')$.

Claim 5.11. The diagram

$$\begin{array}{ccc} MU(n-1) & \xrightarrow{\varphi_{n-1}} & E \\ \downarrow & \nearrow_{\varphi_n} & \\ MU(n) & & \end{array}$$

commutes up to homotopy. ◇

This is the same as checking that the following diagram commutes (up to homotopy)

$$\begin{array}{ccc} MU(0)\otimes MU(n-1) = \mathbb{S}\otimes MU(n-1) & \xrightarrow{\varphi_{n-1} = \varphi_0\varphi_{n-1}} & E \\ \downarrow & & \\ MU(1)\otimes MU(n-1) & \xrightarrow{\varphi_1\varphi_{n-1}} & E \\ \downarrow & & \\ MU(n) & \xrightarrow{\varphi_n} & E \end{array}$$

We have just seen that the lower triangle commutes. Thus we just need to show that $\varphi_1|_{MU(0)} = \varphi_0$. However this follows since this is precisely the condition that the complex orientation $t \in \tilde{E}^2(\mathbb{CP}^\infty)$ restricts to the unit in $\tilde{E}^2(S^2) \cong \pi_0 E$.

The upshot of this discussion is that we get a map $MU \cong \operatorname{colim} MU(n) \to E$. Actually what we have constructed is a bunch of compatible maps in the homotopy category. This is not enough to get a map from the homotopy colimit. What we do have (from the universal property of homotopy colimits) is the equation

$$\operatorname{Map}(\operatorname{hocolim} MU(n), E) = \operatorname{holim} \operatorname{Map}(MU(n), E).$$

This gives an exact sequence

$$\lim{}^1 E^{*-1}(MU(n)) \to E^*(MU) \to \lim E^*(MU(n)) \to \lim{}^1 \cdots .$$

However the map $E^*(MU(n+1)) \to E^*(MU(n))$ is surjective (it is the quotient by the ideal generated by c_{n+1}) so all the $\lim^1$ terms vanish.

Claim 5.12. φ is a map of ring spectra. i.e

$$\begin{array}{ccc} MU \otimes MU & \longrightarrow & MU \\ \downarrow & & \downarrow \\ E \otimes E & \longrightarrow & E \end{array}$$

commutes (up to homotopy). ◇

Proof. We have maps $MU(m) \otimes MU(n) \to MU(m+n)$ which induces a map $MU \otimes MU \to MU$ (this uses another similar obstruction theory argument). Even better, it comes from the map $BU \times BU \to BU$. This is a very good example of a ring spectrum. It is in fact an E_∞-ring spectrum. We won't use that right now. In any event we just saw $E^*(MU \otimes MU) \to \lim E^*(MU(m) \otimes MU(n))$ is an isomorphism. Thus it is enough to check that it commutes for finite n, m, which we have done previously. □

Remark 5.13. MU has a canonical complex orientation. It is a class $t \in \tilde{E}^2(\mathbb{CP}^\infty) = \mathrm{Hom}(MU(1), E)$ such that $\mathbb{S} = MU(0) \to MU(1) \to E$ is the unit map. Now the map $MU(1) \to MU$ is exactly the complex orientation, and $\varphi : MU \to E$ carries this orientation to the orientation of E. ◇

Theorem 5.14. *MU is the universal complex oriented ring spectrum. More precisely for all E, we have a bijection* $\mathrm{Hom}(MU, E) \to \{$*complex orientations of* $E\} \subset E^2(\mathbb{CP}^\infty) = \mathrm{Hom}(MU(1), E)$

Proof. We know it is surjective. Injectivity is proven as follows. Suppose that $\varphi, \varphi' : MU \to E$ give the same complex orientation on E. We want $\varphi \simeq \varphi'$. We know that $\varphi_1 \simeq \varphi_1'$, and $E^*(MU) = \lim E^*(MU(n))$. Thus it suffices to show that $\varphi_n \sim \varphi_n'$ for all n. We use the fact that φ is a ring map. Thus the diagram commutes,

$$\begin{array}{ccc} MU(1)^{\otimes n} & \xrightarrow{\varphi_1^{\otimes n}} & E^{\otimes n} \\ \downarrow & & \downarrow \\ MU(n) & \xrightarrow[\varphi_n,\ \varphi_n']{} & E \end{array}$$

But again we have injectivity $E^*(MU(n)) \to E^*(MU(1)^{\otimes n})$

□

6 Quillen's Theorem

Recall from the last section that if E complex oriented, then we get a graded formal group law $f(x, y)$ over $(\pi_*E)[[x, y]] = E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty)$. If we want to understand these, a good place to start is with the universal example. We saw that the spectrum MU and its universal property. To given a complex orientation of a ring spectrum E is the same as to give a ring spectrum map $MU \to E$. Moreover MU has a canonical universal complex orientation, and a corresponding FGL over π_*MU. By Lazard's theorem, this gives a map of graded rings,

$$\Theta : \mathbb{Z}[t_1, t_2, \cdots] \cong L \to \pi_*MU$$

Theorem 6.1 (Quillen)**.** Θ *is an isomorphism.*

We will prove this theorem in the next few sections. Part of the difficulty is that homotopy groups are hard to compute. Homology is supposed to be easier, so we will start by computing $H_*(MU; \mathbb{Z})$. More generally we can compute $E_*(MU)$ for any complex orientable homology theory. We saw last time that we have isomorphisms

$$E_*(MU) \cong \operatorname{colim} E_*(MU(n)) \cong \operatorname{colim} E_*(BU(n)).$$

Recall that the homology of $BU(n)$ is a free π_*E-module with basis given by symmetric powers of b_i. So $E_*(MU(n) \cong \operatorname{Sym}^n(\pi_*E)\{b_0, b_1, \dots\}$. Where the b_i are the dual basis elements to the t^{i+1}. We have $E^*(MU(1)) \cong t(\pi_*E)[[t]]$. To understand the E-homology of MU we need to understand the how the maps $MU(n) \to MU(n+1)$ behave on E-homology. For any module X, there is always a map

$$\operatorname{Sym}^n X \otimes_{\pi_*E} \operatorname{Sym}^1 X \to \operatorname{Sym}^{n+1} X.$$

We can identify this with some of the structure maps used to define MU. In fact it is the last map of the sequence,

$$E_*(MU(n)) \cong E_*(MU(n) \otimes MU(0)) \to E_*(MU(n) \otimes MU(1)) \to E_*(MU(n+1)).$$

The upshot is that

$$E_*(MU) \cong \operatorname{Sym}^*(\pi_*E)\{b_1, b_1, \dots\} \cong (\pi_*E)[b_1, b_2, \dots].$$

In particular $H_*(MU; \mathbb{Z}) \cong \mathbb{Z}[b_1, b_2, b_3, \dots]$. So by Hurewicz we get a composite map

$$L \to \pi_*MU \to H_*(MU) \cong Z[b_1, b_2, b_3, \dots].$$

This is not the identity but we will see that this composite can be identified with something familiar from before.

Now consider the ring spectrum $E \otimes MU$, whose homotopy groups are $E_*(MU)$. This ring spectrum has two different orientations given by the two different inclusions $E \to E \otimes MU$ and $MU \to E \otimes MU$.

Question 6.2. How are they related?

Recall that a complex orientaion can be describe as a map $MU(1) \to E \otimes MU$ whose restriction happens to be the unit. In this case we have a diagram,

$$S \to MU(1) \rightrightarrows E \otimes MU(1) \to E \otimes MU.$$

Question 6.3. What are the above two maps, precisely?

Note that maps of spectra from $MU(1) \to E \otimes MU(1)$ are the same as maps of E-module spectra from $E \otimes MU(1)$ to itself. Call these maps $\phi_E, \phi_{MU} : E \otimes MU(1) \to E \otimes MU(1)$. The map ϕ_{MU} is easy. It is the identity. The other map factors as a composite,

$$\phi_E : E \otimes MU(1) \to E \simeq E \otimes MU(0) \to E \otimes MU(1).$$

Recall that $E \otimes MU(1) \simeq \oplus_{i \geq 0} Eb_i$ where $E_*(MU(1)) = \pi_* E\{b_0, b_1 \dots\}$. So we can look at this as maps,

$$\phi_E, \phi_{MU} : \oplus Eb_i \to \oplus_i Eb_i.$$

We get a matrix of maps for ϕ_E and ϕ_{MU}. Again ϕ_{MU} is the idenity matrix. The map ϕ_E lands in the b_0 component only. So the matrix is concentrated in the first row. Moreover, recall that the b_i classes are dual to the complex orientation classes t^i. So in fact the matrix for ϕ_E has a one in the $b_1 \to b_1$ component and zeros everywhere else. [[★★★ Why is that again?]]

Note $E^{\mathbb{CP}^\infty}$ acts on $E \otimes MU(1)$, which implies that $E^*(\mathbb{CP}^\infty)$ acts on $E_*(\mathbb{CP}^\infty)$. In particular the element t induces a map $E \otimes MU(1) \to \Sigma^{-2} E \otimes MU(1)$.

$$\oplus Eb_i \xrightarrow{t^i} \oplus Eb_i \to \oplus Eb_i \xrightarrow{b_i} \oplus Eb_i$$

[[★★★ didn't follow this.]] We get a formula

$$\phi_{MU} = \sum_{i \geq 0} b_i \circ \phi_E \circ t^i.$$

So $(E \otimes MU)^*(\mathbb{CP}^\infty) \cong \pi_*(E \otimes MU)[[t]] \cong (\pi_* E)[b_1, b_2, \dots][[t]]$ via ϕ_E. But we also have an isomorphism with $\pi_*(E \otimes MU)[[t']]$ via the map ϕ_{MU}. We have that,

$$t' = \sum b_i t^{i+1}.$$

Upshot: Over $R = \pi_{even}(E \otimes MU) \cong (\pi_* E)[b_1, \dots]$ we have two FGLs $f_E(x,y)$ and $f_{MU}(x,y)$ in $R[[x,y]]$. We learn that $f_{MU}(x,y) = g^{-1} f_E(g(x), g(y))$ where g is the above transformation (or its inverse).

Let's specialize to the case of ordinary cohomology. In that case $R = H_*(MU) = \mathbb{Z}[b_1, b_2, \dots]$, and $f_E(x,y) = x + y$. So $f_{MU} = g^{-1}(g(x) + g(y))$, with

$$g(x) = \sum b_i x^{i+1}.$$

So the map $L \to \pi_* MU \to H_*(MU; \mathbb{Z})$ classifies the formal group law f_{MU} from Section 3. So this composite is *not* an isomorphism, but it is close. We can conclude that Quillen's theorem is true rationally.

We also understand the discrepency between L and $H_*(MU)$ algebraically. What we really need to underatand is why the discrepency between $\pi_* MU$ and $H_*(MU)$ is the same. Later, we will prove this with the Adams spectral sequence. Now we will record a few more corollaries

Corollary 6.4. *$\pi_* MU$ is finitely generated in each degree (since $H_*(MU)$ is finitely generated in each degree).*

To prove Quillen's theorem it suffices to show that $L \otimes \mathbb{Z}_p \to (\pi_* MU) \otimes \mathbb{Z}_p \cong (\pi_* MU)^\wedge_p$ is an isomorphism for all primes p. The homology is similar. It is $\mathbb{F}_p[b_1, b_2, \dots] \cong H_*(MU; \mathbb{F}_p)$.

7 The Adams Spectral Sequence

We want to prove that $L \to \pi_* MU$ is an isomorphism. To prove this we need to compute $\pi_* MU$. Last time we saw how to compute $H_*(MU, \mathbb{F}_p) \cong \mathbb{F}_p[b_1, \dots]$

we can think of these as $\pi_*(\mathbb{S} \otimes MU)$ and $\pi_*(H\mathbb{F}_p \otimes MU)$, induced by the map of (E_∞) ring spectra $\mathbb{S} \to H\mathbb{F}_p$. Think of this as a map $\operatorname{Spec} H\mathbb{F}_p \to \operatorname{Spec} \mathbb{S}$. MU is a module for S, so it is like a sheaf, which we have pulled back to $\operatorname{Spec} H\mathbb{F}_p$. If we want to recover information about the original sheaf we would consider descent on $\operatorname{Spec} H\mathbb{F}_p \times_{\operatorname{Spec} \mathbb{S}} \operatorname{Spec} H\mathbb{F}_p = \operatorname{Spec} H\mathbb{F}_p \otimes_S H\mathbb{F}_p$, and the higher fiber products. This is what the Adams spectral sequence does.

$H\mathbb{F}_p$ is a ring spectrum. And it gives a cosimplicial spectrum $R^\bullet$, with $R^n = (H\mathbb{F}_p)^{\otimes n+1}$. it is coaugmentented with a map from $\mathbb{S}$. Given any spectrum X we can smash this whole gadget. This gives a map,

$$X \to Tot(X \otimes R^\bullet)$$

where the totalization is the homotopy limit. In good cases this map is "nice". In particular any connective spectrum whose homotopy groups are finitely generated abelian groups, then it induced an isomorphism in mod p cohomology. It is the mod p completion of X.

In this case there is a spectral sequence $\{E_r^{a,b}, d_r\}$ such that $E_1^{a,b} = \pi_a X \otimes R^b$, and the first differential is the alternating sum of the cosimplicial face maps. This also tells us what the E_2-term is.

Theorem 7.1 (Adams)**.** $X \to Tot(X \otimes R^\bullet) \simeq \operatorname{holim} Tot^n(X \otimes R^\bullet) \to Tot^n(X \otimes R^\bullet)$. *This defines a filtration* $\cdots \subset F^2\pi_n X \subset F^1 \pi_n X \subset F^0 \pi_n X = \pi_n X$, *where* $F^i \pi_n X = \ker(\pi_n X \to \pi_n Tot^{i-1} X)$. *Then this filtration is comensurate with the p-adic filtration, i.e. for all i there exists* $j >> i$ *such that* $p^j \pi_n X \subset F^j \pi_n X \subset p^i \pi_n X$. *In particular* $F^b \pi_{a-b} X / F^{b+1} \pi_{a-b} X \cong E_\infty^{a,b}$. *and*

$$(\pi_n X)^\wedge \cong \lim \pi_n X / F^i \pi_n X.$$

$E_1^{a,0} = H_a(X; \mathbb{F}_p)$. and $E_1^{a,1} \cong \pi_a(X \otimes H\mathbb{F}_p \otimes H\mathbb{F}_p)$.

We write $X \otimes H\mathbb{F}_p \otimes H\mathbb{F}_p = (X \otimes H\mathbb{F}_p) \otimes_{H\mathbb{F}_p} H\mathbb{F}_p \otimes H\mathbb{F}_p$ and so the homology is $H_*(X; \mathbb{F}_p) \otimes \pi_*(H\mathbb{F}_p \otimes H\mathbb{F}_p) = H_*(H\mathbb{F}_p; \mathbb{F}_p) = \mathcal{A}^\wedge$, the dual of the Steenrod algebra.

So we see that $E_1^{a,1} \cong H_*(X; \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathcal{A}^\wedge$. Similarly we have

$E_1^{*,2} \cong H_*(X; \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathcal{A}^\wedge \otimes_{\mathbb{F}_p} \mathcal{A}^\wedge$.

So if $V = H_*(X; \mathbb{F}_p)$, then we have a cosimplicial object,

$$V \rightrightarrows V \otimes_{\mathbb{F}_p} \mathcal{A}^\wedge \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} V \otimes_{\mathbb{F}_p} \mathcal{A}^\wedge \otimes_{\mathbb{F}_p} \mathcal{A}^\wedge \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}}$$

What are the differentials? Well $\mathcal{A}^\wedge$ is a Hopf algbera, induced by the map $H\mathbb{F}_p \otimes H\mathbb{F}_p = H\mathbb{F}_p \otimes \mathbb{S} \otimes H\mathbb{F}_p \to H\mathbb{F}_p \otimes H\mathbb{F}_p \otimes H\mathbb{F}_p$.

So in the language of algebraic geometry we have $\operatorname{Spec} \mathcal{A}^\wedge \to \operatorname{Spec} H\mathbb{F}_p$ is a group object over $\operatorname{Spec} H\mathbb{F}_p$. We've just see the multiplication map. In fact it is associative and has a unit. We'll see in a minute what group it is.

The first map $V \otimes_{\mathbb{F}_p} \mathcal{A}^\wedge \to V \otimes_{\mathbb{F}_p} \mathcal{A}^\wedge \otimes_{\mathbb{F}_p} \mathcal{A}^\wedge$ is given by a map $V \to V \otimes_{\mathbb{F}_p} \mathcal{A}^\wedge$ is given by a map $X \otimes \mathbb{S} \otimes H\mathbb{F}_p \to X \otimes H\mathbb{F}_p \otimes H\mathbb{F}_p$.

We get the complex,

$V \to \Gamma(G; V \otimes \mathcal{O}_G) \to \Gamma(G \times G; V \otimes \mathcal{O}_{G \times G}) \to \cdots$

conclusion $G = \operatorname{Spec} \mathcal{A}^\wedge$ acts on $H_*(X; \mathbb{F}_p)$ and $E_2^{*,b} \cong H^b(G; V) \cong \operatorname{Ext}^b_{\mathcal{A}^\wedge - \mathrm{comod}}(\mathbb{F}_2, V)$, cohomology of the group G in the module V.

Question 7.2. What does this group G look like?

For simplicity we will assume $p = 2$. (In that case we need to use super alegbraic geometry since everything is graded commutative). Look for a representation of G. Let $X_n = \mathbb{S}^{\mathbb{RP}^n} = \mathbb{D}(\Sigma^\infty_+ \mathbb{RP}^n)$. Then $H_*(X_n; \mathbb{F}_2) \cong H^*(\mathbb{RP}^n; \mathbb{F}_2) \cong \mathbb{F}_2[t]/(t^{n+1})$. This is true for all n, so G acts on the direct limit of the $V_n = \operatorname{Spec} X_n$. This is algebraic object $\operatorname{Spf} \mathbb{F}_2[[t]]$, formal scheme of the ring $H^*(\mathbb{RP}^\infty; \mathbb{F}_2)$.

Just as in the case of complex line bundles, there is a map $\mathbb{RP}^\infty \times \mathbb{RP}^\infty \to \mathbb{RP}^\infty$ which classifies tensoring line bundles. This gives a FGL over $\mathbb{F}_2$. I.e. it gives a group structure on $\operatorname{Spf} H^*(\mathbb{RP}^\infty; \mathbb{F}_2)$. In this case it is $f(x, y) = x + y$. So we get $G \to Aut(\operatorname{Spf} \mathbb{F}_2[[t]])$. but it factors through those power series which preserve the group structure, i.e. $g(t + t') = g(t) + g(t')$. This means that

$$g(t) = a_o t + a_1 t^2 + a_2 t^4 + \cdots .$$

Theorem 7.3 (Milnor)**.** *This map induces an isomorphism from G to the group of transformation of this form in which $a_0 = 1$. i.e. as a scheme G is an affine space of infinite dimension.*

This is a paraphrasing of Milnor's theorem.

——— [[★★★ merge this together]]

G is a group scheme over $\mathbb{F}_2$. This means that for every $\mathbb{F}_2$-algebra R we can form $G(R) = \operatorname{Hom}(\operatorname{Spec} R, G) = \operatorname{Hom}(\mathcal{A}^\wedge, R)$.

Looking at the Spanier-Whitehead dual of $\mathbb{RP}^n$ we see that G acts on $\operatorname{Spec} H^*(\mathbb{RP}^n; \mathbb{F}_2)$, i.e. $\mathcal{A}^\wedge$ co-acts on $H^*(\mathbb{RP}^n; \mathbb{F}_2)$. This is given by a map,

$$g : t \mapsto a_1 t + a_2 t^2 + \cdots + a_n t^n$$

These are compatible for different n and so we get an un-truncated power series action of G. This is an action which comes from geometry, so it is going to preserve any structure we have on these rings coming from topology. One such structure is the formal group law $f(x, y) = x+y \in R[[x, y]]$ coming from the multiplication in $\mathbb{RP}^\infty$. This means that $g(x+y) = g(x) + g(y)$. This seems unlikely, but remember we are in characteristic 2 [[★★★ What happens in characteristics other than 2? this argument with $\mathbb{RP}^\infty$ seems special to this prime.]]

Theorem 7.4 (Milnor)**.** *This map induces an isomorphism from G to the group of transformation of the form*

$$g(t) = a_o t + a_1 t^2 + a_2 t^4 + \cdots .$$

in which $a_0 = 1$. i.e. as a scheme G is an affine space of infinite dimension.

[[★★★ (look at Jacob's notes on course on Steenrod algebra to see a proof of this).]]

Special case $X = MU$.

$H_*(MU; \mathbb{F}_2) \cong \mathbb{F}_2[b_1, b_2, \dots]$. $\operatorname{Spec} H_*(MU; \mathbb{F}_2) \cong \mathbb{A}^\infty$ over $\mathbb{F}_2$. We have a formal group law $f(x, y) = x + y \in R[[x, y]]$. This gives rise to R-points which look like formal expressions,

$$g(x) = x + b_1 x^2 + b_2 x^3 + \cdots$$

in $R[[x]]$. The R-points are the coordinate changes of Spf $R[[x]]$ which agree with the coordinate x up to first order.

G acts on $\operatorname{Spec} H_*(MU;\mathbb{F}_2)$ = coordinates on $\operatorname{Spf}\mathbb{F}_2[[x]]$ which agree with x up to first order. Moreover G is the automorphisms of $\operatorname{Spf}\mathbb{F}_2[[y]] = \operatorname{Spf} H^*(\mathbb{RP}^\infty;\mathbb{F}_2)$ as a formal group, which are the identity up to first order. There is an obvious action, but *This Is Not The Action!*. These identifications come from different parts of topology. The generator y has degree 1 and x has degree 2 ($\mathbb{F}_2[[x]] = H^*(\mathbb{CP}^\infty;\mathbb{F}_2)$.

There is a map $\mathbb{RP}^\infty \to \mathbb{CP}^\infty$ which is complexification. So we get $H^*(\mathbb{CP}^\infty) \to H^*(\mathbb{RP}^\infty)$, which sends y to x^2 (this is a homomorphism since we are in characteristic 2).

This means that $g(y)$ in G acts on $R[[x]]$ as

$$g(x) = x + a_1^2x^2 + a_2^2x^4 + \cdots$$

Question 7.5. How does G act on $\operatorname{Spec} H_*(MU;\mathbb{F}_2)$?

Solution: We look at the Frobenius $F : G \to G = G'$ (we name the target G' to keep track of it), then we act using the obvious action.

So we get a sequence

$$K \to G \xrightarrow{F} G'$$

Where the kernel K consists of those series $g(t) = t + a_1t^2 + a_2t^3 + \cdots$ such that $a_i^2 = 0$.

$$E_2^{*,b} = H^b(G; H_*(MU;\mathbb{F}_2)) = H^b([\operatorname{Spec} H_*(MU;\mathbb{F}_2)/G])$$

Let $Z = \operatorname{Spec} H_*(MU;\mathbb{F}_2) = \mathbb{A}^\infty$. We want to understand the stack Z/G. We first understand Z/G'.

Claim 7.6. G' acts freely on Z. ◇

Proof. let $Z_0 \subseteq Z$ be $\operatorname{Spec}\mathbb{F}_2[b_i]$ which $i+1$ is not a power of 2. $Z_0 = \operatorname{Spec}\mathbb{F}_2[b_2, b_4, b_5, b_6, b_8, \dots]$. This receives a map from $\mathbb{Z}[b_1, b_2, b_3, \dots]$ by sending the relevant b_is to zero. □

Observe that the restriction of the action map $G' \times Z_0 \to Z$ is an isomorphism. i.e. any expression $g(x) = x + b_1x^2 + b_2x^3 + \dots$ has a factorization $g = g' \circ g''$ where $g'(x) = x + b_2'x^3 + b_4'x^5 + \cdots$ and $g''(x) = x + a_1x^2 + a_2x^4 + \cdots$.

The proof is done by inductively solving for the coordinates. You can see that at each stage there is exactly a linear equation whose solution is the new coefficient.

Corollary 7.7. $[Z/G'] \cong Z_0 \cong \mathbb{A}^\infty$.

We really want $[Z/G] = [[Z/K]/G'] = [[Z \times BK]/G']$ [[★★★ The map $Z_0 \times BK \times G' \to Z \times BK$ is an isomorphism so that $Z/G = Z_0 \times BK$. This explain incorrectly in the notes!]]

So we get

$$E_2^{*,b} \cong H^b(Z/G) \cong \mathbb{F}_2[b_2, b_4, b_5, b_6, b_8, \dots] \otimes_{\mathbb{F}_2} H^b(K;\mathbb{F}_2)$$

But $K(R) = \{g(y) = y + a_1y^2 + a_2y^4 + \dots \mid a_i^2 = 0\}$ is abelian.

[[★★★ calculation....]] $K(R) = \prod_{i>0}\alpha_2$ where $\alpha_2(R) = \{a \in R \mid a^2 = 0\}$.

More explicitly, $\alpha_2 = \operatorname{Spec}\mathbb{F}_2[a]/(a^2)$, with coalgebra structure $\mathbb{F}_2[a]/(a^2) \to \mathbb{F}_2[a_1, a_2]/(a_1^2, a_2^2)$ given by $a \mapsto a_1 + a_2$. Note: As a coalgebra $\mathbb{F}_2[a]/(a^2)$ is isomorphic to $\mathbb{F}_2^{\mathbb{Z}/2}$, i.e the ring of functions on $\mathbb{Z}/2$. This is not an isomorphism of algebras, just coalgebras.

However the cohomology of an affine group scheme = Hopf algbebra never uses the algebra structure, just the coalgebra structure. So we see that $H^*(\alpha_2) = H^*(\mathbb{Z}/2; \mathbb{F}_2) = H^*(\mathbb{RP}^\infty; \mathbb{F}_2) = \mathbb{F}_2[[\epsilon]]$. So we have,

$$E_2^{*,b} \cong \mathbb{F}_2[b_i, \epsilon_j]$$

where $i+1 \neq$ a power of 2, and $j \geq 1$. Moreover the bidegree of b_i is $(2(i-1), 0)$, and the bidegree of ϵ_j is $(2^j - 1, 1)$.

this is because $K(R) = \{g(y) = y + a_1 y^2 + a_2 y^4 + \dots \mid a_i^2 = 0\}$ which are automorphisms of $H^*(\mathbb{RP}^\infty)$. This means y has degree -1 and a_j has degree $2^j - 1$.

The total degree of b_i is $2(i-1)$ and the total degree of ϵ_j is $2^j - 2$.

Note that both of these are even. This means that (at the prime 2) the Adams spectral sequence of MU degenerates at the second page.

Claim 7.8. For $X = MU$ and $H\mathbb{F}_p$, the Adams Sepctral Seqeunce has $E_2^{0,b} \cong \mathbb{F}_p[b_i, \varepsilon_j]$ with $i+1 \neq p^n$ and all j, with $|b_i| = (2i, 0)$ and $|\varepsilon_j| = (2p^j - 1, 1) \mapsto 2(p^j - 1)$. ◇

We proved this last time in the case $p = 2$.

Let $c_i = b_i$ if $i+1 \neq p^n$ or ε_j if $i+1 = p^j$. Then c_i is in total degree $2i$. These are all in even degrees so

$$E_\infty^{a,b} \cong E_2^{a,b} \cong \mathbb{F}_p[c_0, c_1, c_2, \dots]$$

So the graded $gr\pi_* MU \cong \mathbb{F}_p[c_0, c_1, \dots]$. Without loss of generality, c_0 is represented by $p \in \pi_0 MU$. $F^1\pi_0 MU = \ker(\mathbb{Z} \to \mathbb{Z}/p) = p\mathbb{Z}$.

Choose for each $i > 0$ a class $x_i \in \pi_{2i} MU$ such that

– if $i+1 \neq p^n$, then x_i represents c_i,
– if $i+1 = p^j$, then $x_i \in F^1\pi_* MU$ and represents c_i.

We get a map $\mathbb{Z}[x_1, x_2, \dots] \to \pi_* MU$ of graded commutative rings. Define $F^b\mathbb{Z}[x_i]$ to be the subgroup generated by the monomials $p^{m_1} x_1^{m_2} x_2^{m_3} \cdots$ such that $m_1 + m_2 + \cdots \geq b$. Then the map $\mathbb{Z}[x_i] \to \pi_* MU$ is a map of filtered rings. Note that $gr\mathbb{Z}[x_i] \to gr\pi_* MU \cong \mathbb{F}_p[c_i]$ is an isomorphism. Consequence: $\mathbb{Z}[x_i]/F^b\mathbb{Z}[x_i] \cong \pi_* MU/F^b\pi_* MU$, and so we get,

$$\mathbb{Z}_p[x_1, x_2, \dots] \cong \lim \mathbb{Z}[x_i]/F^b\mathbb{Z}[x_i] \cong \lim \pi_* MU/F^b\pi_* MU \cong \pi_* MU \otimes_{\mathbb{Z}} \mathbb{Z}_p$$

Recall that we had a map $L \xrightarrow{\theta} \pi_* MU \to H_*(MU; \mathbb{Z}) \cong \mathbb{Z}[b_1, b_2, \dots]$. We know that it is injective. It is enough to prove θ is surective after p-adic completion at every prime p. So we have:

$$\mathbb{Z}_p[t_1, t_2, \dots] \to \mathbb{Z}_p[x_1, x_2, \dots] \to \mathbb{Z}_p[b_1, b_2, \dots]$$

There are ideals I, K, J of the elements of positive degree.

$$(I/I^2)_{2n} \to (K/K^2)_{2n} \to (J/J^2)_{2n}$$

$$\mathbb{Z}_p t_n \xrightarrow{\lambda} \mathbb{Z}_p x_n \xrightarrow{\lambda'} \mathbb{Z}_p b_n$$

We want λ to be invertible (so the first map is surjective). We know that $\lambda\lambda' = 1$ if $n+1 \neq p^v$ and is p if $n+1 = p^v$. So we are okay in the first case. [[★★★ Then some argument I missed which shows it works which when $p+1 = p^v$.]]

8 The Adams-Novikov Spectral Sequence

Want to understand $\pi_* X$, given $E_* X$. There are two extremes: when $E = \mathbb{S}$ in which this is trivial and the usual Adams spectral sequence when $E = H\mathbb{F}_p$. The mod p homology of X is "easy", but the spectral sequence is difficult. In general E is in the middle between $\mathbb{S}$ and $H\mathbb{F}_p$. What we would like to balance these tensions so that we can compute the E-homology, but where the spectral sequence is not too difficult. The case we will now consider is $E = MU$.

We have the coaugmented cosimplicial spectrum,

$$X^* = X \to X \otimes MU \rightrightarrows X \otimes MU \otimes MU \Rrightarrow X \otimes MU^{\otimes 3} \cdots$$

This gives rise to a spectral sequence known as the *Adama-Novikov Spectral Sequence* computing the totalization. In this case $X \to Tot(X^*)$ is an equivalence if X is connective. We get a filtration by the partial totalizations and these give the spectral sequence.

Again if X is connective, then $F^b\pi_{a+b}X/F^{b+1}\pi_{a+b}X \cong E_\infty^{a,b}$, and $E_1^{a,b} = \pi_a(X \otimes MU^{\otimes b+1}$. We would like to understand what the d_1 differential is so that we can understand what the E_2-term is. It will be some sort of (possible unfamiliar) homology/cohomology.

Look at the portion,

$$MU_*(X) \rightrightarrows (MU \otimes MU)_*(X)$$

This second term is $\pi_*((MU \otimes MU) \otimes_{MU} (MU \otimes X))$ and $\pi_*(MU \otimes MU) \cong (\pi_* MU)[b_1, b_2, \dots] = L[b_1, b_2, \dots]$. This has two formal group laws, one the usual one on L, and the other given by a change of coordiantes $g(x) = x + b_1x^2 + b_2x^3 + \dots$.

Thus we get $\oplus_\alpha \Sigma^d MU \to MU \otimes MU$ is an isomorphism. Hence $MU \otimes MU$ is 'free' over MU. This means that $(MU \otimes MU)_*(X) \cong MU_*(X) \otimes_{\mathbb{Z}} \mathbb{Z}[b_1, b_2, \dots]$.

of course there are more terms in the cosimplicial spectrum, but each term can be computed in a similar way. For example the next term is $MU_*(X) \otimes_{\mathbb{Z}} \mathbb{Z}[b_1, b_2, \dots] \otimes_{\mathbb{Z}} \mathbb{Z}[b_1, b_2, \dots]$.

Let $G = \operatorname{Spec} \mathbb{Z}[b_1, b_2, \dots]$ so that $G(R) = \{g(t) = t + b_1t^2 + \dots \in R[[t]]\}$ is a group scheme (these power series compose). Then $\operatorname{Spec} L(R) \cong FLG(R)$ is acted on by $G(R)$. Hence G acts on $\operatorname{Spec} L$. So the above sequence becomes

$$\operatorname{Spec} L \leftleftarrows \operatorname{Spec} L \times G \;\overset{\leftarrow}{\underset{\leftarrow}{\leftleftarrows}}\; \operatorname{Spec} L \times G \times G \;\overset{\leftarrow}{\underset{\leftarrow}{\leftleftarrows}}\; \cdots$$

If X is a spectrum, the smash this with X. Conclusion: For any X, $MU_*(X)$ is a module over $\pi_* MU \cong L$. Moreover, this module is equipped with an action of G, compatible with the action of G on L.

And furthermore, the above complex is exactly the thing computing

$$E_2^{*,*} \cong H^*(G; MU_*(X)) \cong H^*([\operatorname{Spec} L/G]; X).$$

where $[\operatorname{Spec} L/G]$ is the quotient stack.

9 The Moduli Stack of Formal Groups

Recall that we were looking at the stack $\operatorname{Spec}(L)/G$ which is FGL mod $G(R) = R[[t]]$. Now if X was a spectrum, then $MU_*(X)$ is a module for $L = \pi_* MU$. It also has an action of G which is compatible with G's action on L and we get a spectral sequence,

$$E_2^{*,b} = H^b(G; MU_*(X))$$

the Adams-Novikov spectral sequence. Se we define the stack $\mathcal{M}_{FG}^s = \operatorname{Spec}[L/G]$. This E_2-term is the cohomology of this stack with values in a coherent sheaf $\mathcal{F}_X$. Now we have been a little vague because there is another grading (besides b). This is a 'geometric' grading and comes from an action of $\mathbb{G}_m$ on $MU_*(X)$ (at least the even part). The action of G doesn't commute with this action.

$$\begin{aligned} G &= \operatorname{Spec} \mathbb{Z}[b_1, b_2, b_3, \dots] \\ \mathbb{G}_m &= \operatorname{Spec} \mathbb{Z}[b_0^{\pm 1}] \\ G^+ &= \operatorname{Spec} \mathbb{Z}[b_0^{pm1}, b_1, b_2, b_3, \dots] \end{aligned}$$

and G^+ is the semi-direct product $G \ltimes \mathbb{G}_m$. Moreover, G^+ acts on L (there is no reason the power series substitution needs to start with 1, any unit will do).

The module $MU_*(X)$ is a representation of G^+ which is compatible with its action on L. So we have a coherent sheaf over the stack $\mathcal{M}_{FG} = \operatorname{Spec} L/G^+$. There is a quotient map $\mathcal{M}^s_{FG} \to \mathcal{M}_{FG}$.

Let R ba a commutative ring. let $Alg_R =$ the category of comm. R-algebras. Then we define

$$\begin{aligned} \operatorname{Spf} R[[t]] : Alg_R &\to \mathsf{Set} \\ A &\mapsto \operatorname{Hom}_{cts}(R[[t]], A) = \{a \in A \mid a \text{ is nilpotent}\} \end{aligned}$$

Let $f \in R[[x, y]]$ be a formal group law over R. Then we can define a functor $\mathcal{F}_f : Alg_R \to \mathsf{Ab}$ by

$$\mathcal{F}_f(A) = \{a \in A \mid a \text{ is nilpotent}\}$$

with the group structure $(a, b) \mapsto f(a, b)$ (which makes sense because a and b are nilpotent.

Definition 9.1. A *coordinatizable formal group over* R is a functor $\mathcal{F} : Alg_R \to \mathsf{Ab}$ such that $\mathcal{F} \simeq \mathcal{F}_f$ for some $f \in R[[x, y]]$. ◇

I.e. we have forgotten the specific formal group law and just remembered the functor. Suppose that f and f' are FGLS over R. When are $\mathcal{F}_f$ and $\mathcal{F}_{f'}$ isomorphic? The data of an automorphism $\alpha : \mathcal{F}_f \to \mathcal{F}_{f'}$ is an automorphism α_A of $\{a \in A \mid a \text{ is nilpotent}\}$ for each A.

For simplicity assume that R has no nilpotents, and let $A = R[t]/(t^n)$. Then the radical of A is $\sqrt{A} = tA$. The map $\alpha : \sqrt{A} \to \sqrt{A}$ and so we have,

$$t \mapsto b_0 t + b_1 t^2 + \cdots + b_{n-1} t^n.$$

Moreover these coeffiecients don't depend on n. We can see this by looking at $A' = R[t]/(t^{n+1})$ and comparing via the map $A' \to A$. Thus we can conclude that there is a power series,

$$g(t) = b_0 t + b_1 t^2 + \cdots$$

(with b_0 invertible) such that α_A is given by g for all A.

Remark 9.2. If you don't want to assume that R has no nilpotents, then you just need to think of $\operatorname{Spf} R[[t]]$ as a scheme with a distinguished point (the unit zero). You look at automorphisms which preserve this point. ◇

Now if $f, f' \in R[[x, y]]$, then α gives an isomorphism of $\mathcal{F}_f$ and $\mathcal{F}'_f$ precisely when

$$f'(g(x), g(y)) = g(f(x, y)).$$

We call this an *isomorphism* of formal group laws. Now the stack $\mathcal{M}_{FG} = [\operatorname{Spec} L/G^+]$ can (in a preliminary sense) be thought of as a functor $\mathcal{M}_{FG} : comm \to Grpd$, sending a comm. ring to the category of formal group laws over R with isomorphism of formal group laws as morphisms.

The reason that this is preliminary is that as written it does not satisfy descent for any of the usual algebraic topologies. The reason is that the coordinates which exist locally might not glue up to global coordinates (even though the formal groups do glue up).

Definition 9.3. Let R be a commutative ring. A functor $\mathcal{F} : Alg_R \to \mathsf{Ab}$ is a *formal group* over R if

1. $\mathcal{F}$ is a sheaf (in the Zariski topology) $x + y = 1 \in A$ implies
2. it is locally a coordinatizable formal group.

◇

Definition 9.4. Let $\mathcal{F}$ be a formal group over R. The *Lie algebra* of $\mathcal{G}$

$$\mathfrak{g}_{\mathcal{F}} = \ker(\mathcal{F}(\mathbb{R}[t]/(t^2)) \to \mathcal{F}(R)).$$

◇

If $\lambda \in R$, then $t \mapsto \lambda t$ induces a map $\mathbb{R}[t]/(t^2) \to \mathbb{R}[t]/(t^2)$ and so gives a map

$$\lambda : \mathfrak{g}_{\mathcal{F}} \to \mathfrak{g}_{\mathcal{F}}.$$

Moreover $\mathfrak{g}_{\mathcal{F}} = tR[t]/t^2 \cong R$ with the group structure of addition in R, and the map multiplication by λ is additive in λ [[★★★ I might of missed something here]]. We conclude that $\mathfrak{g}_{\mathcal{F}} \cong R$ as an R-module.

In general $\mathfrak{g}_{\mathcal{F}}$ is an invertible R-module of projective rank one.

Claim 9.5. If $\mathcal{F}$ is a formal group, then $\mathcal{F}$ is coordinatizable if and only if $\mathfrak{g}_{\mathcal{F}} \cong R$. ◇

[[★★★ Draw diagram of three intersecting neighborhoods. They have formal group laws on the sets and power series on the overlaps.]]

This gives a class in $H^1(\operatorname{Spec} R; G^+)$. There is an exact sequence,

$$H^1(\operatorname{Spec} R; G) \to H^1(\operatorname{Spec} R; G^+) \to H^1(\operatorname{Spec} R; \mathbb{G}_m).$$

But $H^1(\operatorname{Spec} R; G)$ is trivial because G admits a filtration with successive quotients the additive group (the additive group is a quasi-coherent sheaf and its cohomology vanishes) and so (say be a spectral sequence) we see that $H^1(\operatorname{Spec} R; G) = 0$. In contrast, the group $H^1(\operatorname{Spec} R; \mathbb{G}_m)$ classifies line bundles.

Our goal for the next part of the course will be to understand the stack $\mathcal{M}_{FG}$.

$\mathcal{M}_{FG}(R)$ = the category of formal groups over R with isomorphisms as morphisms. There is a fiber sequence,

$$\mathbb{G}_m \to \mathcal{M}^s_{FG} \to \mathcal{M}_{FG}$$

If $\mathcal{F}$ is a formal group over R, then we can assign to $\mathcal{F}$ its Lie algebra $\mathfrak{g}$which is a projective R-module of rank one. $\mathcal{F}$ is coordinatizable if and only if this module is free. Consider the assignment,

$$(R, \mathcal{F}) \mapsto \mathfrak{g}^{-1}$$

(where $\mathfrak{g}^{-1}$ is the inverse bundle to $\mathfrak{g}$. This is a quasi-coherent sheaf, in fact a line bundle ω on $\mathcal{M}_{FG}$. The stack $\mathcal{M}^s_{FG} = \omega \smallsetminus \{0\}$ is the total space minus the zero section.

Why do we care?If X is a spectrum, then $MU_*(X)$ is an L-module with a compatible action of G^+. Then the Adams-Novikov spectral sequence has

$$E_2^{*,b} \cong H^b(G; MU_*(X)) \cong H^b(\mathcal{M}^s_{FG}; \mathcal{F}^s)$$

where $\mathcal{F}^s = MU_*(X)$ viewed as a quasi-coherent sheaf on $\mathcal{M}_{FG}^s$. In particular we have

$$E^{0,b} = H^b(\mathcal{M}_{FG}, \mathcal{F})$$

and more generally,

$$E^{2a,b} = H^b(\mathcal{M}_{FG}, \mathcal{F} \otimes \omega^{\otimes a}).$$

Our goal is to understand the stack $\mathcal{M}_{FG}$. In characteristic zero $\mathcal{M}_{FG} \cong B\mathbb{G}_m$ and $\mathcal{M}_{FG}^s \cong pt$. i.e. if R is of characteristic zero, for every formal group law $f(x,y) = g(g^{-1}(x) + g^{-1}(y))$ can be conjugated to the additive group. For example the multiplicative formal group $f_m = x + y + xy$ is given by using

$$g(t) = e^1 - 1 = t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \cdots .$$

This suggest that in positive characteristic these formal group laws are different.

We will define an invariant which will help us tell these apart. Consider $R[[t]]$. A (continuous) differential is an element of the free module $R[[t]]dt$.

Example 9.6. $g(t) = a_0 + a_1 t + \cdots$ and $dg = (a_1 + 2a_2 t + 3a_2 t^2 + \cdots)dt$. ◇

Suppose that $f(x,y)$ is a power series in $R[[x,y]]$. Then

$$f^*(g(t)dt) = (g \circ f) \cdot \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \right)$$

in $R[[x,y]]\{dx, dy\}$. We say that $g(t)dt$ is *invariant* if $f^*(g(t)dt) = g(x)dx + g(y)dy$.

Lemma 9.7. *There is a unique left invariant differential form ω on $\mathbb{G} = \operatorname{Spf} R[[t]]$. such that*

$$\omega = (1 + \cdots)dt.$$

All others are given by $\lambda\omega$ for $\lambda \in R$.

In particular $(R[[t]]dt)^{inv} =$ the fiber of ω on $\mathcal{M}_{FG}$.

Definition 9.8. If f is a FGL over R, an endomorphism of f is a power series $h(t) = b_0 t + b_1 t^2 + \cdots$ such that $hf(x,y) = f(h(x), h(y))$. ◇

Note if ω is left invariant, then $h^*(\omega)$ is left invariant, hence $b_0\omega$ [[★★★ is invariant?]]. $R[[t]]dt = R[[t]]\omega$, then $h^*(g(t)\omega) = (g \circ h)(t) h^*(\omega) = b_0 (g \circ h)(t)\omega$.

Corollary 9.9. *If $b_0 = 0$, then $h^* = 0$. i.e. for all $g(t)$ $h^*(dt) = h'(t)dt = 0$.*

Assume that R is of characteristic p. (i.e. $p = 0$ in R. Then

$$h(t) = a_1 t^p + a_2 t^{2p} + a_3 t^{3p} + \cdots$$

Then $h_0(t) = h(t^p)$ is an endomorphism of another formal group law $f_0(x,y)$ which satisfies,

$$f_0(x^p, y^p) = f(x,y)^p.$$

Examples of endomorphisms. $h(t) = t$ is an endomorphism (the identity). More generally for all $n \geq 0$ we get an endomorphism $[n]$ such that

$$\begin{aligned}[0](t) &= 0\\ [1](t) &= t\\ [n](t) &= f([n-1](t), t)\end{aligned}$$

We have $[n](t) = n{\cdot}t+$ higher order terms. Conclusion: if R is of char p, then $[p](t) = 0{\cdot}t+\cdots$ (higher order terms). $h(t) = ct^m + \cdots$ with $p|m$. Note that

$$[p](t) = h(t^p)$$

where h is a power series endomorphism of a different formal group law. If $m \geq 1$ we can repeat this.

Conclusion:

$$[p](t) = \begin{cases} 0 \\ ct^{p^n} + \text{higher order terms} \end{cases}$$

In the first case we say $f(x, y)$ has height ∞, in the second case we say that f has height $\geq n$. (f has height exactly n if c is invertible).

Example 9.10. $f(x, y) = x+y$ and $[p](t) = pt = 0$ if R has char p. So this has height ∞. ◇

Example 9.11. $f(x, y) = x + y + xy) = (x + 1)(y + 1) - 1$. This $[n](t) = (t + 1)^n - 1$, and in characteristic p we have $[p](t) = t^p$. So the multiplicative formal group law has height one. ◇

Note if we have two formal group laws $f, f' \in FGL(R)$ such that f and f' are isomorphic, then their heights are equal. Since f and f' are isomorphic we have a $g(t)$ such that

$$g(f(x, y)) = f'(g(x), g(y))$$

Hence $g([p]_f(t)) = [p]_{f'}(g(t))$, but since the leading term of g is invertible we have

$$[p]_f = cb_0^{p^n-1}t^{p^n} + \cdots$$

where $[p]_{f'} = ct^{p^n} + \cdots$. This shows that the height in an invariant of the formal group law.

next time we will prove the theorem

Theorem 9.12 (Lazard)**.** *Let k be an algebraic closed field of characteristic p, the two FGLs are isomorphic if and only if they have the same height. Moreover every height is realized.*

Definition 9.13. f is a FGL over R. Then v_n is th coefficient of tp^n in $[p](t)$. Note that f has height $\geq n$ is equivalent to $v_0, v_1, \cdots, v_{n-1}$ are zero. $v_0 = p$. ◇

Really we should think of $v_0, v_1, \ldots$ as elements of the Lazard ring L. Recall that the isomorphism $L \cong \mathbb{Z}[t_1, t_2, \ldots]$ was not canonical, but the isomorphisms $(I/I^2)_{2n}j \cong \mathbb{Z}\{t_n\}$ was canonical.

Claim 9.14. In L we have $v_n \in (p^{p^n-1} - 1)t_{p^n-1} + I^2$. ◇

These v_n are precisely the generators of L which don't go to generators of $\mathbb{Z}[b_1, b_2, \dots]$. They are the generators in Adams filtration one.

Recall,

$$[p](t) = pt + \cdots + v_1 t^p + \cdots + v_2 t^{p^2} + \cdots$$

f has height $\geq n$ if $v_0 = v_1 = \cdots v_{n-1} = 0$ and height exactly one if in addition $v_n \in \mathbb{R}^\times$ is invertible.

We had an isomorphism $L \cong \mathbb{Z}[t_1, t_2, \dots]$ which was not canonical, but the isomorphisms $(I/I^2)_{2n} \cong \mathbb{Z}\{t_n\}$ was canonical. Now the claim is that $v_n \in L_{2(p^n-1)}$ maps to $(p^{p^n-1}-1)t_{p-1}$ plus decomposables, where t_{p-1} is the generator of $(I/I^2)_{2(p^n-1)}$.

Look at

$$L \to \mathbb{Z} \oplus (I/I^2)_{2(p^n-1)} \cong \mathbb{Z} \oplus \mathbb{Z}t$$

classified by the FGL

$$\begin{aligned} f(x,y) &= x + y + t \sum_{0<i<p^n} \frac{\binom{p^n}{i}}{p} x^i y^{p^2-i} \\ &= x + y + \frac{t}{p}((x+y)^{p^n} - x^{p^n} - y^{p^n}) \end{aligned}$$

so we have (over the rationals)

$$[a](x) = ax + \frac{t}{p}((ax)^{p^n} - ax^{p^n})$$

hence

$$[p](x) = px + \frac{t}{p}(p^{p^n} - p)x^{p^n}$$

Corollary 9.15. *Let k be a field of characteristic p. For all $n \geq 1$, there exists a formal group law over k of height n.*

Proof. Choose any map $L \to k$ such that $t_i \mapsto 0$ if $i < p^n - 1$, and $t_{p^n-1} \mapsto 1$. □

Remark 9.16. The condition that $f \in R[[x,y]]$ has height $\geq n$ is Zariski local. i.e. if we have $R \to R'$ and a FGL over R of height $\geq n$, then so is the corresponding FGL over R'. This means that this concept makes sense for formal groups (which don't necessarily have a FGL, but do locally). ◇

Definition 9.17. $\mathcal{M}_{FG}^{\geq n}$ is the moduli stack of formal groups of height $\geq n$. ◇

$\mathcal{M}_{FG}^{\geq n}$ is a closed algebriac substack of $\mathcal{M}_{FG}$.

Note the ideal $(v_0, v_1, \dots, v_n)$ is G^+-invariant, but the v_i are not individually G^+-invariant. We have $\mathcal{M}_{FG}^{\geq n} = [\mathcal{S}pec(L/(v_0, v_1, \dots, v_{n-1})/G^+]$. Not that $\mathcal{M}_{FG}^{\geq 1} = \mathcal{M}_{FG} \times_{\mathcal{S}pec\,\mathbb{Z}} \operatorname{Spec} \mathbb{F}_p$. Although the v_i are not individually invariant, the element v_n is G-invariant (not G^+) not in L, but in $L/(v_0, v_1, \dots, v_{n-1})$. Hence it is acted on by $G^+/G = \mathbb{G}_m$ by a character. This character is $(p^n - 1)id$. Hence v_n is not a function on $\mathcal{M}_{FG}$, nor on $\mathcal{M}_{FG}^{\geq n}$, but it is a *section* over $\mathcal{M}_{FG}^{\geq n}$ of ω^{p^n-1}.

Definition 9.18. $\mathcal{M}_{FG}^{n} = \mathcal{M}_{FG}^{\geq n} \smallsetminus \mathcal{M}_{FG}^{\geq n+1}$ ◇

We have $\mathcal{M}^n_{FG} = [(\mathrm{Spec}(L/(v_0, \ldots, v_{n-1})[v_n^{-1}])/G^+]$. We also define $\mathcal{M}^\infty_{FG} = \cap_n \mathcal{M}^{\geq n}_{FG}$.
We can localize $\mathcal{M}_{FG} \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\mathbb{Z}_{(p)}$, then we have

– $\mathcal{M}^0_{FG} \cong B\mathbb{G}_m$

–

Claim 9.19. $f \in R[[x, y]]$ has height infinity if and only if f is isomorphic to a group law $x + y$. (and $p = 0$ in R. ◇

Converse: Let $p = 0$ in R. $f(x, y)$ has height $\geq n$ if and only if f is isomorphic to the FGL f' such that [[★★★ erased before noted]].

Proof. By induction on $m < p^n$, we construct $f_m(x, y) = g_m^{-1}(f(g_m(x), g_m(y))$ such that $f_m(x, y) = x + y$ modulo $(x + y)^m$.

Lemma 9.20. *Let $f(x, y), f'(x, y) \in R[[x, y]]$ be such that $f \equiv f'$ moudlo $(x + y)^m$. Let $g = gcd\{\binom{}{mi}\}_{0 \leq i < m} = p$ if $m = p^n$ and is 1 otherwise. Then there exists a unique $\mu \in R$ such that*

$$f(x, y) \equiv f'(x, y) + \mu \sum_{0<i<m} \frac{\binom{m}{i}}{d} x^i y^{m-i}$$

modulo $(x + y)^m$.

[[★★★ .... proof of lemma in third lecture.....]]

□

Goal choose an isomorphic FGL f' such that $f'(x, y) = x + y \bmod (x + y)^{m+1}$.
Two cases m not a power of p, then $x + y + \frac{\mu}{d} \sum \ldots$ and

$$g(x) = x \pm \frac{\mu}{d} x^m.$$

The problem is when m is a power of p.
claim: $f(x, y) = x + y$ modulo $(x + y)^{m+1}$.
$\phi : L \to R$ sending t_i to zero for $i < m-1$. What is $\phi(t_{m-1})$? if is zero since $\phi(t_{m-1} = -v_n$. So we get a sequence of FGLs $f_1, f_2, f_3 \ldots$ equivalent to f.

Claim 9.21. This also works for $n = \infty$. ◇

The power series g_m are converging to a power series in the t-adic topology and the allows us to define f_∞.
So we have this stratification of $\mathcal{M}_{FG}$ and we understand the ∞ and zero height cases. We now want to understand $\mathcal{M}^n_{FG}$ when $0 < n < \infty$.

Claim 9.22. There is a unique *geometric* point, i.e. if $k = \overline{k}$ of characteristic p, then any two formal group laws over k are isomorphic if and only if they have the same height. ◇

Theorem 9.23 (Lazard)**.** *Let $f, f' \in FGL(k)$ be formal group laws of the same height, n. If $k = \overline{k}$, then $f \cong f'$.*

We will actually prove a much stronger result, but to state it now would be a pedagogical disaster.

Proof. WLOG we have $f'(x,y) \equiv x + y + \lambda \sum_{0<i<p^n} \frac{\binom{p^n}{i}}{p} x^i y^{p^n-i}$, mod $(x,y)^{p^n+q}$. Let's fix some notation. Let

$$\chi_m(x,y) = \sum_{0<i<m} \frac{\binom{m}{i}}{d} x^i y^{m-i} \tag{9.24}$$

where $d = \gcd\left(\binom{m}{i}\right) = \ell$ if $m = \ell^k$ and is 1 otherwise.

Strategy: Define a sequence of FGLs $f_1(x,y) = f(x,y), f_2(x,y), f_3(x,y)$ such that

1. $f_m(x,y) = g_m^{-1} f_{m-1}(g_m(x), g_m(y))$ for some $g_m(t) = b_0 t + \cdots$
2. $f_m(x,y) = f'(x,y)$ mod $(x,y)^{m+1}$,
3. $g_m(t) \to t$ in the t-adic topology (hence $g = g_2 \circ g_3 \circ g_4 \circ \cdots$ is the desires isomorphism.)

Assume that $f_{m-1}(x,y)$ is defined. Then $f_{m-1}(x,y) \equiv f'(x,y) + \mu\chi_m(x,y)$ mod $(x,y)^{m+1}$.

Case 1: m is not a power of p. Then $g_m(t) = t + ct^m$, and $f_m(x,y) = g_m^{-1} f_{m-1}(g_m(x), g_m(y))$. Then,

$$\begin{aligned} f_m(x,y) &\equiv f_{m-1}(x,y) + c \sum_{0<i<m} \frac{\binom{m}{i}}{d} x^i y^{m-i} \\ & f_{m-1}(x,y) - dc\chi_m(x,y). \end{aligned}$$

This works if $\mu = -dc$, where then $c = -\frac{\mu}{d}$ which makes sense if $m \neq p^n$.

Case 2: $m = p^{n'}$ with $n' < n$. Then we claim $f(x,y) \equiv f'(x,y)$ mod $(x,y)^{m+1}$, and so $f_{m-1}(x,y) \equiv x + y$ mod $(x,y)^{p^{n+1}}$.

Case 3: $m = p^n$. Then $f_{m-1}(x,y) \equiv f'(x,y) \equiv x + y$ mod $(x,y)^{p^n}$. And we have

$$\begin{aligned} f_{m-1}(x,y) &\equiv x + y + \lambda\chi_m(x,y) \\ f'(x,y) &\equiv x + y + \lambda'\chi_m(x,y) \end{aligned}$$

mod $(x,y)^{p^n+1}$ with $\lambda, \lambda' \neq 0$.

We need to solve $c^{p^n-1}\lambda = \lambda'$ $c^{p^n-1} = \frac{\lambda}{\lambda'}$ [[★★★ missed this argument, we multiply by a scalar.]]

Case 4: $m = p^{n+n'}$ with $n' > 0$. Then $f_{m-1}(x,y) = f'(x,y) + \mu\chi_m(x,y)$ mod $(x,y)^{m+1}$. Idea is tho use $g_m(t) = f_{m-1}(t, ct^{p^{n'}})$. Notation: f a FGL, then $f(x,y,z) = f(f(x,y),z)$ and $f(w,x,y,z) = f(w, f(x, f(y,z)))$. Since f is associative, these are unambiguous. There are obvious generalizations of these.

With the above g we get $g_m f_m(x,y) = f_m(x, y, cx^{p^{n'}}, cy^{p^{n'}})$. Now let $z = z(x,y)$ be such that $cf_m(x,y)^{p^n} = f_{m-1}(z, cx^{p^{n'}}, cy^{p^{n'}})$. We learn that

$$f_{m-1}(f_m(x,y), z) = f_{m-1}(x,y) \tag{9.25}$$

Claim:

1. $z \equiv 0$ mod $(x,y)^m$ and
2. $f_m(x,y) \equiv f_{m-1}(x,y) \equiv f'(x,y)$ mod $(x,y)^m$.

Similarly we have claims:

$a_{m'}$ $z \equiv 0 \bmod (x,y)^m$ and

$b_{m'}$ $f_m(x,y) \equiv f_{m-1}(x,y) \equiv f'(x,y) \bmod (x,y)^m$.

where we claim these for $m' \leq m$.

Proof. Induction on m': $(a_{m'}) \Rightarrow (b_{m'})$. Assume that $m' < m$, $(a_{m'})$ and $(b_{m'})$ hold, then we want to prove $(a_{m'+1})$.

[[★★★ didn't quite follow this argument, go back and review it.There are two congruence expressions, which we subtract to get something congruent to something.]] This gives us

$$f_m(x,y) \equiv f_{m-1}(x,y) + z \tag{9.26}$$

mod $(x,y)^{m+1}$. Question: What is z modulo $(x,y)^{m+1}$? Well $f_{m-1}(x,y) \equiv x + y + \lambda\chi_{p^n}(x,y)$ mod $(x,y)^{p^n+1}$ and so we have [[★★★ modifies previous expression]]. [[★★★ more and more]]

□

$Isom(f,f') = \operatorname{Spec} R'$ where $R' = \operatorname{colim} R = R(0) \to R(1) \to R(2) \to \cdots$ where each $R(i-1) \to R(i)$ is finte étale.

□

[[★★★ Can get the explicit radius of convergence too.]]

E a complex-oriented cohomology theory, then $R = \pi_{even}E$ is a commutative ring. $\mathbb{G} = \operatorname{Spf} E^{even}(\mathbb{CP}^\infty)$ a formal group over R with coordinate. $L \to R$, with L the Lazard ring $L = \mathbb{Z}[t_0, t_1, \dots]$.

Question 9.27. Given R and a formal group law over R can we build an E?

Idea: Define $E_*(X) = MU_*(X) \otimes_L R$ (there is a similar formula in cohomology if X is a finite complex). Does this give a homology theory? Answer: No in general. Because $\otimes_L R$ need not preserve exact sequence (and you need the long exact Mayer-Vietoris sequences). So we learn that the answer is "yes" if R is flat over L. However, L is very large and so this is a very strong condition.

However, we can do better. Recall that we have:

$$\begin{array}{ccc} \operatorname{Spec} R & \longrightarrow & \operatorname{Spec} L \\ & \overset{\phi}{\searrow} & \downarrow \\ & & \mathcal{M}_{FG} \end{array}$$

What we actually need is that ϕ be flat.

Definition 9.28. A *quasi-coherent sheaf* on $\mathcal{M}_{FG}$ is a rule which assigns to $\eta \in \mathcal{M}_{FG}(R)$ an R-module $F(\eta)$ which is functorial in R (plus associativity) ◇

i.e. for $R \to R'$ with $\eta \mapsto \eta' \in \mathcal{M}_{FG}(R')$, then $F(\eta) = F(\eta') \otimes_R R'$. Think $\mathcal{F}(\eta) = \eta^*\mathcal{F}$ and $\eta : \operatorname{Spec} R \to \mathcal{M}_{FG}$.

Example 9.29. If X is a spectrum, then we get a quasi-coherent $\mathcal{F}_X$ where $\mathcal{F}_X(\eta_0) = MU_{even}(X)$, where $\eta_0 \in \mathcal{M}_{FG}(L)$. (Recall that $\mathcal{M}_{FG} = \operatorname{Spec} L/G^+$). ◇

Question 9.30 (Kirsten). How do we see this example?

Answer:The quickest way is to look at the Adams-Novikov resolution for the spectrum X. This gives all the data for a coherent sheaf on $\mathcal{M}_{FG}^s$, then we look at gradings to get the object over $\mathcal{M}_{FG}$.

Definition 9.31. $\mathcal{F}$ is *flat* if for all $\eta \in \mathcal{M}_{FG}(R)$ we have $\eta^*\mathcal{F}$ is a flat R-module. $\diamond$

Note that flatness is local so it suffices to test when η is cooridinatizable. $L \to R$. And $\mathcal{F}(\eta) = \mathcal{F}(\eta_0) \otimes_L R$.

$$\begin{array}{ccc} \operatorname{Spec} B & \hookrightarrow & \operatorname{Spec} R' \\ \downarrow & & \downarrow \eta \\ \operatorname{Spec} R & \xrightarrow[\eta']{} & \mathcal{M}_{FG} \end{array}$$

M is flat over $\mathcal{M}_{FG}$ if $\eta_* M$ is flat in the sense of the previous definition.

Claim 9.32. Let M be an L-module which is flat over $\mathcal{M}_{FG}$. Then the functor

$$\begin{aligned} QCoh(\mathcal{M}_{FG}) &\to (L\text{-Modules}) \\ \mathcal{F} &\mapsto (\eta^*\mathcal{F}) \otimes_L M \end{aligned}$$

is exact. More, generally let Let M be an R-module which is flat over $\mathcal{M}_{FG}$. Then the functor

$$\begin{aligned} QCoh(\mathcal{M}_{FG}) &\to (R\text{-Modules}) \\ \mathcal{F} &\mapsto (\eta^*\mathcal{F}) \otimes_R M \end{aligned}$$

is exact. $\diamond$

It is enough to check flatness for the base change,

$$\begin{array}{ccc} \operatorname{Spec} R[b_0^{\pm 1}, b_1, b_2, \ldots] & \hookrightarrow & \operatorname{Spec} R \\ \downarrow & & \downarrow \\ \operatorname{Spec} L & \xrightarrow[\eta']{} & \mathcal{M}_{FG} \end{array}$$

$$(\mathcal{F}(\eta') \otimes_L R[b_0^{\pm 1}, b_1, b_2, \ldots]) \otimes_{R[b_0^{\pm 1}, b_1, b_2, \ldots]} M[b_0^{\pm 1}, b_1, b_2, \ldots] \tag{9.33}$$

But this is isomorphic to

$$\mathcal{F}(\eta') \otimes_L M[b_0^{\pm 1}, b_1, b_2, \ldots] \tag{9.34}$$

and $M[b_0^{\pm 1}, b_1, b_2, \ldots]$ is a flat L-module.

Corollary 9.35. *If M is a graded L-module which is flat over $\mathcal{M}_{FG}$, then $X \to MU_*(X) \otimes LM$ is a homology theory.*

Question 9.36. Given an L-module M, when is M flat over $\mathcal{M}_{FG}$?

Theorem 9.37 (landweber)**.** *This is true if and only if for every prime p the sequence, $v_0 = p$, v_1, v_2, $\dots$ is a regular sequence on M.*

Proof. next time □

Recall that if R is a commutative ring with $x_0, x_1, x_2, \dots \in R$, and M an R-module, then $x_0, x_1, x_2, \dots$ is M-regular if

$$\begin{aligned} 0 \to & M \xrightarrow{x_0} M \to M/x_0 \to 0 \\ 0 \to & M/x_0 \xrightarrow{x_1} M/x_0 \to M/(x_0, x_1) \to 0 \\ 0 \to & M/(x_0, x_1) \xrightarrow{x_2} M/(x_0, x_1) \to M/(x_0, x_1, x_2) \to 0 \\ & \dots \end{aligned}$$

Example 9.38. $R = \mathbb{Z}[\beta, \beta^{-1}]$ with the degree of β equal to 2, then $f(x, y) = x + y + \beta xy$, and we have $[n](t) = \frac{1}{\beta}[(1+\beta t)^n - 1]$, and $[p](t) \equiv \beta^{p-1} t^p \bmod p$. Fix p, then $v_0 \mapsto p \in R$, and $v_1 \mapsto \beta^{p-1} \pmod{v_0}$, and $v_1 \in (R/v_0)^\times$. We conclude that Landweber's criterion is satisfied and we get a homology theory E_*. In fact this is K-theory, $E \simeq K$. ◇

Example 9.39. R a commutative ring, and $E \to \operatorname{Spec} R$ an elliptic curve over R. Then we get a formal group,

$$\hat{E}(A) = \{????\} \tag{9.40}$$

[[★★★ if the Lie algebra of E is a free A-module]] ◇

To satisfy Landweber's criterion we need for all primes p that p is a non-zero divisor in R and v_1 in R/p a non-zero divisor.

If f has height exactly n, then we have

$$\operatorname{Spf} R[[t]]/[p](t) \to \operatorname{Spf} R[[t]] \xrightarrow{p} \operatorname{Spf} R[[t]] \tag{9.41}$$

so that the kernel has rank p^n.

For elliptic curves, v_1 is a function which vanishes on the locus of supersingular elliptic curves. (the Hasse invariant). So Landweber's implies criterion is that

If R is torsion free and the Hesse invariant of E is a non-zero divisor on R/p for all p, then we get a graded formal group law over $R[\beta, \beta^{-1}]$ which satisfies Landweber's criterion. Then we get a cohomology theory E with $\pi_* E \cong R[\beta, \beta^{-1}]$.

(without the Lie algebra condition we get a spectrum with $\pi_* E = \oplus \mathfrak{g}^{\otimes n}$).

Theorem 9.42 (Exact Functor Theorem)**.** *An L-module M is flat over $\mathcal{M}_{\mathrm{FG}}$ if and only if for every prime p the sequence $p = v_0, v_1, \dots$ is M-regular.*

In the proof we will use the following special case:

Example 9.43. $R = L_{(p)}/(t_i)_{i+1 \neq p^k} \cong \mathbb{Z}_{(p)}[v_1, v_2, v_3, \dots]$. ◇

$$\begin{array}{ccc} \operatorname{Spec} R[b_0^{\pm 1}, b_1, b_2, \dots] & \hookrightarrow & \operatorname{Spec} L \\ \downarrow \ulcorner & & \downarrow \\ \operatorname{Spec} R \longrightarrow \operatorname{Spec} L & \longrightarrow & \mathcal{M}_{\mathrm{FG}} \end{array}$$

The thing in the upper left corner it is $L_{(p)}[b_0^{\pm 1}, b_1, b_2, \dots]/(\text{images } t_i)_{i+1 \neq p^k}$. This maps to $\mathbb{Z}_{(p)}[b_0^{\pm 1}, b_1, b_2, \dots]$ (coming from the map $L_{(p)} \to \mathbb{Z}_{(p)}$).

we also have the following diagram

$$\begin{array}{ccc} \operatorname{Spec} L_{(p)}[b_0^{\pm 1}, b_1, b_2, \dots] & \hookrightarrow & \operatorname{Spec} L \\ \downarrow \ulcorner & & \downarrow \\ \operatorname{Spec} \mathbb{Z}_{(p)}[v_1, \dots] & \xrightarrow[\text{faithfully flat}]{} & \mathcal{M}_{\mathrm{FG}} \times \operatorname{Spec} \mathbb{Z}_{(p)} \end{array}$$

So to prove Landweber's criterion it suffices to prove it at every prime. Let M be an L-module and p a prime. Let $\operatorname{Spec} B$ be this last pull-back. Then the claim is that the sequence $p = v_0, v_1, v_2$ is M-regular if and only if $M_B := M \otimes_L B$ is flat over $\mathbb{Z}_{(p)}[v_1, v_2, \dots]$.

What does it mean for M_B to be flat over $\mathbb{Z}_{(p)}[v_1, v_2, \dots]$? It means that for ever $\mathbb{Z}_{(p)}[v_1, v_2, \dots]$-module N, $Tor_i^{\mathbb{Z}_{(p)}[v_1, v_2, \dots]}(M_B, N) = 0$ for $i > 0$. This means we can assume that N is finitely presented (since any N is a filtered colimit of these). Without loss of generality we can assume that $N = N_0[v_{n+1}, v_{n+2}, \dots]$ for some module N_0 over $\mathbb{Z}_{(p)}[v_1, v_2, \dots, v_n]$. Then this tor group reduces to

$$Tor_i^{\mathbb{Z}_{(p)}[v_1, v_2, \dots, v_n]}(M_B, N). \tag{9.44}$$

algebraic digression: Let R be any commutative ring and M and R-module. $x \in R$ is not a zero divisor. Then M is flat if and only if

1. x is a non-zero divisor on M,
2. M/x is flat over R/x,
3. $M[x^{-1}]$ is flat over $R[x^{-1}]$.

Proof. Want ${}_i^R(M, N)$ vanishes fro all R-modules. Step 1: if $x : N \to N$ is zero.

$$Tor_i^R(M, N) \cong Tor^{R/x}(M/x, N) \cong 0 \tag{9.45}$$

The first isomorphism is by (1) and the second is by (2). Step 2 is that $Tor_i^R(M, N)$ vanishes if $x^k : N \to N$ is zero. Step 3: This is true if N is x-power torsion, $N \to N[x^{-1}]$ is zero. This is because N is the direct limit of module for which the Tor's vanish. Step 4: Let $N_0 \subset N$ be the kernel of $N \to N[x^{-1}]$, and consider the exact sequence,

$$0 \to N_0 \to N \to N/N_0 \to 0 \tag{9.46}$$

By (3) $Tor_i^R(M, N_0) = 0$ for $i > 0$. So to prove that $Tor_i^R(M, N) = 0$, it suffices to prove the $Tor_i^R(M, N/N_0) = 0$, i.e. we've reduced to the case where x is a non-zero divisor on N. Step 5: Consider

$$0 \to N \to N[x^{-1}] \to N[x^{-1}]/N \to 0 \tag{9.47}$$

By assumption (3), the group $Tor_{i+1}^R(M, N[x^{-1}]/N) \cong 0$. To prove $Tor_i^R(M, N) \cong 0$ we need $0 \cong Tor_i^R(M, N[x^{-1}]) \cong Tor_i^{R[x^{-1}]}(m[x^{-1}], N[x^{-1}])$ [[★★★ missed next sentence.]] □

So we have two maps $\phi', \phi'' : L \to B$

$$\begin{array}{ccc} \operatorname{Spec} B & \xhookrightarrow{\phi''} & \operatorname{Spec} L \\ {\scriptstyle \text{from } \phi'}\downarrow & \ulcorner & \downarrow \\ \operatorname{Spec} \mathbb{Z}_{(p)}[v_1, \dots] & \longrightarrow & \mathcal{M}_{\text{FG}} \times \operatorname{Spec} \mathbb{Z}_{(p)} \end{array}$$

Let $v_i'' = \phi''(v_i) \in B$, and $v_i' = \phi'(v_i) \in B$. Note that for $m \geq 0$, the sequences $v_0', v_1', \dots, v_{m-1}'$ and $v_0'', v_1'', \dots, v_{m-1}''$ generate an ideal $I_m \subseteq B$. We have $v_m' \equiv \lambda v_m''$ mod I_m for λ invertible.

Claim 9.48. for all $0 \leq m \leq n+1$ $M_B/I_m M_B$ is flat over $\mathbb{Z}_{(p)}[v_1, \dots, v_n]/(v_0, \dots, v_{m-1})$. ◇

The claim implies the theorem for $m = 0$. What do we know so far? We know the claim is true for $m = n + 1$, because the ring is $\mathbb{Z}_{(p)}[v_1, \dots, v_n]/(v_0, \dots, v_n) \cong \mathbb{F}_p$ is a field (all modules are flat over it).

Proof. Desending induction on m. Assume $m \leq n$. Claim holds for $m + 1$, want the claim for m. $R_{m+1} \cong R_m/(v_m)$. We have a sequence

$$0 \to M_B/I_m M_B \stackrel{v_m'}{\to} M_B/I_m M_B \to M_B/I_m M_B \tag{9.49}$$

[[★★★ I missed why this was injective. B is flat over L?]]. We want $M_B/I_m M_B$ is flat over R_m. By the lemma it suffices to check that

1. v_m is not a zero divisor,
2. $(M_B/I_m M_B)/v_m$ is flat over R_m/v_m,
3. $(M_B/I_m M_B)[v_m^{-1}]$ is flat over $R_m[v_m^{-1}]$.

□

[[★★★ another big commutative diagram]] It suffices to shw that $(M_B/I_m M_B)[v_m^{-1}]$ is flat over $\operatorname{Spec}(\mathbb{Z}_{(p)}[v_1, \dots]/(v_0, \dots, v_{m-1}))[v_m^{-1}]$.

$$\operatorname{Spec}(\mathbb{Z}_{(p)}[v_1, \dots]/(v_0, \dots, v_{m-1}))[v_m^{-1}] \to \mathcal{M}_{\text{FG}}{}^m \to \mathcal{M}_{\text{FG}} \tag{9.50}$$

An even better statement is that (localized at p) every quasi-coherent sheaf on $\mathcal{M}_{\text{FG}}{}^m$ is flat.

$$\begin{array}{ccc} \operatorname{Spec} R' & \longrightarrow & \operatorname{Spec} R \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathbb{F}_p & \longrightarrow & \mathcal{M}_{\text{FG}}{}^m \end{array}$$

Proof. R' is the universal R-algebra which classifies isomorphisms f with f', where f and f' are formal group laws clasified by maps $\operatorname{Spec} R \to \mathcal{M}_{\text{FG}}{}^m$ and $\operatorname{Spec} \mathbb{F}_p \to \mathcal{M}_{\text{FG}}{}^m$ (not all such maps are given by FGLs (they are formal groups), but locally they are given by FGLs). This means that R' is the colimit of [[★★★ missed it.]] [[★★★ The map $\operatorname{Spec} R' \to \operatorname{Spec} R$ is faithfully flat]] □

Morally, the proof of this statement is that ${\mathcal{M}_{\text{FG}}}^m$ has only one point. ${\mathcal{M}_{\text{FG}}}^m \cong$ Spec $/\overline{F}_p$/Morava stabilizer group. (We'll see this later.)

In anyevent, we get a cohomology theory associated to $\mathbb{Z}_{(p)}[v_1, v_2, \dots]$. This is known as Brown-Peterson theory $BP_*(X) \cong MU_*(X)_{(p)}/(t_i)_{i+1neqp^k}$.

Recall the following theorem (Adam's version of Brown's representation theorem)

Theorem 9.51.

– *Let h_* be a cohomology theory. Then there exists a spectrum E such that $h_*(X) \cong E_*(X)$.*

– *Let E, F be spectra, any map of homology theories $E_* \to F_*$ is given by a map $E \to F$.*

what this doesn't tell you is that the map is unique. In fact it is not, not even up to homotopy. This means that E is not really canonical even in the homotopy category of spectra. But earlier we were looking at $E_*(X) = MU_*(X) \otimes_L E_*$. How unique is this spectrum?

Definition 9.52. A map of spectra $f : E \to E'$ is a *phantom map* if it induces $0 : E_*(X) \to E'_*(X)$ for all spaces X. ◇

The ambiguity in the spectra maps above is by phantom maps.

Lemma 9.53. *let $f : E \to E'$ be a map of spectra. TFAE*

1. *f is phantom*
2. *f is zero on $E_*(X) \to E'_*(X)$ for all spectra X*
3. *Same for all finite spectra X.*
4. *For all finite X, $E^*(X) \to (E')^*(X)$ is zero.*
5. *For all finite X, with $X \to E$, the map $X \to E \to E'$ is null.*

Proof. The equivalence of (1) and (2) is given by the fact that $X = \text{colim}(\sigma^{\infty-1}\Omega^{\infty-n}X)$. Similarly any spectrum is a the colimit of finite spectra. For finite spectra we can look at the alexander Whitney dual to get (3) equivalent to (4), and (4) and (5) are equivalent. □

Definition 9.54. A spectrum E is Landweber-exact if there exists an evenly graded L-module M such that

1. M satisfies Landweber's criterion: for all p, the sequence $v_0 = p, v_1, \dots$ is M-regular.
2. $E_*(X) \cong MU_*(X) \otimes_L M$.

◇

Theorem 9.55 (Hopkins?)**.** *If E, E' are Landweber exact, then every phantom map $f : E \to E'$ is null-homotopic.*

Definition 9.56. A finite spectrum X is *even* is $H_n(X, \mathbb{Z})$ are free abelian groups and zero in n is odd. (Equivalently X has a cell decomposition using only even cells). ◇

Definition 9.57. A spectrum E is evenly generated if for every finite X, every map $X \to E$ factors through a finite even spectrum. ◇

A fintite spectrum is evenly generated if it is even.

Theorem 9.58. *Let E be an evenely generated. Let E' be a spectrum with $\pi_n E' = 0$ for odd n. Then every phantom map $E \to E'$ is a null-homotopy.*

Lemma 9.59. *E is Landweber-exact implies that E is Evenly generated.*

Proof of Theorem 9.58. Let A be the set of equivalence classes of maps $X_\alpha \to E$ for X_α an even finite spectrum. Then we have

$$K \to \oplus_\alpha X_\alpha \to E$$

Then the map $E \to \Sigma K$ is a phantom map. It is a "versal" phantom map. Any phantom map factors through it, i.e a phantom $E \to E'$, factors as $E \to \Sigma K \to E'$. So it suffices to prove that any map $\Sigma K \to E'$ is null.

Even better, we will show that K is a retract of a sum of even finite spectra, $\oplus_\beta Y_\beta$. This would show that

$$(E')^{-1}(K)(E')^{-1}(\oplus Y_\beta) = \prod (E')^{-1}(Y_\beta) = 0$$

the last equality follws by the AHSS.

Now to construct the Y_β, we look at B, the collection of all homotopy classes of maps $X_\alpha \xrightarrow{g} X\alpha' \to E$. Set $Y_\beta = X_\alpha$. Then we have a map $(id, g) : Y_\beta \to X_\alpha \oplus X_{\alpha'}$. Taking the difference gives a map

$$\oplus Y_\beta \to \oplus X_\alpha$$

$$\begin{array}{ccc} \oplus Y_\beta & \hookrightarrow & \oplus X_\alpha \\ \downarrow & & \downarrow \\ K & \longrightarrow & \oplus X_\alpha \end{array}$$

So the cofiber of the top is F and we have a map $F \to E$. Next idea, construct $q : E \to F$. By Adams, we need to map $E_* \to F_*$, i.e. we need to construct a map $X \to F$, for all maps $X \to E$ with X finite.

We have a map $E \to F \to E$, which we don't know much about, but we do know that it is a homotopy equivalnce. This induces a map $K \to \oplus Y_\beta \to K$, which is also a homotopy equivalence. Hence K is a retract of Y. □

of Lemma 9.59. We have that M is an evenly graded L-module, $E_* \cong MU_* \otimes_L M$. Let X be finite and $\eta : X \to E$. Equivalently $\eta \in E_0(DX) \cong (MU_*(DX) \otimes_L M)_0$, so $\eta = \sum a_i \otimes m_i$, where $a_i \in MU_{-d_i}(DX)$ and $m_i \in M_{d_i}$. Note: all d_i are even (since the m_i are.) So we have,

$$X\{a_i\} \to \oplus \sum^{d_i} MU\{m_i\} \to E$$

to prove that E is evenly generated suffices to show that $\oplus \sum^{d_i} MU$ is evenly generated, i.e. need to show that MU is evenly generated.

Now $MU = \operatorname{colim} MU(n)$, so it suffices to look at each $MU(n)$. The are the (desuspensions) of Thom spaces $BU(n)^{\xi_n}$. So it suffices to look at the $BU(n)$. Each $BU(n) = \operatorname{colim} Gr(n, \mathbb{C}^{n+m})$, so it suffices to look at the Thom spaces of the Grassmanians. Now these have an (algebraic) decomposition (given by the Bruhat decomposition) into even dimensional cells. □

Next week: Morava Stablizer groups, Morava E-theory, and Morava K-theory.

10 Answering Some questions...

Recall if we are given a map of even graded commutative rings $u : L \to R$ which satisfies Landweber's criterion, then we get a spectrum E with $E_*(X) \cong MU_*(X) \otimes_L R$

1. What can we do if we don't have a coordinate, but just a formal group?
2. What about the gradings?
3. How canonical is this?

If $G \to \operatorname{Spec} R$ is a formal group, then it has a Lie algeabra $\mathfrak{g}$ which is an invertible module.

Definition 10.1. R is a commutative ring, and $\mathcal{L}$ is an invertible R-module (proj. rank one), then an *$\mathcal{L}$-twisted FGL* is an expression $f(x, y) = \sum a_{i,j} x^i y^j$ satisfying the expressions for identity, commutativity , and associativity. However the coefficients are sections $a_{i,j} \in \mathcal{L}^{\otimes i+j-1}$ [[★★★ check these lie in the correct spaces]]. ◇

Note, if we can trivialize $\mathcal{L}$ and do so, then this is equivalent to an ordinary group law.

If f is an $\mathcal{L}$-twisted FGL, we get a formal group $\mathcal{F}_f(A) = \{R\text{-linear maps} \mathcal{L} \to nil(A)\}$ (landing in nilpotent elements of A). The expression f gives the group structure. Moreover, there is a canonical isomorphism $\mathfrak{g}_{\mathcal{F}_f} \cong \mathcal{L}^{-1}$, where the left-hand side is the Lie algebra of $\mathcal{F}_f$. This is the same as the kernel of

$$\mathcal{F}_f(R[\varepsilon]/(\varepsilon^2)) \to \mathcal{F}_f(R).$$

Claim 10.2. Let $\mathcal{F}$ be any formal group over R, with Lie algebra $\mathfrak{g}$. Then there exists a $\mathfrak{g}^{-1}$-twisted FGL f, and an isomorphism $\mathcal{F} \cong \mathcal{F}_f$, such that the isomorphism,

$$\mathfrak{g} = \mathfrak{lie}(\mathcal{F}) \cong \mathfrak{lie}(\mathcal{F}_f) \cong \mathfrak{g}$$

is the identity. ◇

The proof is like last time [[★★★ See Jacob's notes.]]

Note an $\mathcal{L}$-twisted FGL over R is the same as a graded FGL over $\oplus_{n\in\mathbb{Z}} \mathcal{L}^n$ (where $\mathcal{L}$ is in degree 2). If $\mathcal{L}$ is trivial, then we have $\oplus\mathcal{L} \cong R[\beta, \beta^{-1}]$, which is also equivalent to a graded ring homomorphism $L \to \mathcal{L}$.

Question 10.3. Why do we take $\mathbb{Z}$-grading and not just positive integers?

Later we will need these extra inverses to show Landweber exactness. (e.g. connective K-theory is not Landweber exact, but periodic K-theory is.)

Claim 10.4. [[★★★ Might be dubious]] Let f be an $\mathcal{L}$-twisted FGL. Then we get a digram as below, and the following are equivalent:

$$\begin{array}{ccc} \operatorname{Spec} \oplus \mathcal{L}^n & \longrightarrow & \operatorname{Spec} L \\ \downarrow & \searrow^{q'} & \downarrow \\ \operatorname{Spec} R & \xrightarrow[q]{} & \mathcal{M}_{\mathrm{FG}} \end{array}$$

1. q is flat
2. q' is flat
3. $\oplus\mathcal{L}^n$ satisfies Landweber's criterion.

◇

If these conditions are satisfied, then we get a spectrum E_R (which a priori depends on the choice of $\mathcal{L}$-twisted formal group law) with $(E_R)_*(X) = MU_*(X) \otimes_L (\oplus\mathcal{L}^n)$.

Example 10.5. $R = L$, and $\mathcal{L} = R$. Then E_R is *periodic* complex bordism (written MP). $MP = MU[\beta, \beta^{-1}]$. Recall that MU is the Thom spectrum of universal virtual complex vector over BU of virtual dimension zero. Similarly MP is the Thom spectrum over $BU \times \mathbb{Z}$ for all virtual vector bundles of al virtual dimensions. ◇

Example 10.6. R arbitrary, $\mathcal{L} = R$, the we get a spectrum E_R with $(E_R)_*(X) = MP_*(X) \otimes_L R$

$$\begin{array}{ccc} & & \operatorname{Spec} R \\ & \nearrow & \downarrow q \\ \operatorname{Spec} L & \xrightarrow[p]{} & \mathcal{M}_{\mathrm{FG}} \end{array}$$

Recall, for any spectrum X, there is a quasi-coherent sheaf $\mathcal{F}_X$ on $\mathcal{M}_{\mathrm{FG}}$ such that $p^*\mathcal{F}_X \cong MU_{even}(X) \cong MP_0(X)^{even}$. So we see that $(E_R)_0(X) \cong p^*\mathcal{F}_X \otimes_L R \cong q^*\mathcal{F}_X$. ◇

As a corollary we see that E_R only depends on $q : \operatorname{Spec} R \to \mathcal{M}_{\mathrm{FG}}$ (E_R is well defined in the homotopy category of spectra.)

So any affine chart on the moduli stack of formal groups gives a spectrum. How do these play together? Given $q : \operatorname{Spec} R \to \mathcal{M}_{\mathrm{FG}}$ and $q' : \operatorname{Spec} R' \to \mathcal{M}_{\mathrm{FG}}$, what is $E_R \otimes E'_R$?

$$\begin{array}{ccc} \operatorname{Spec} B & \xrightarrow{q'} & \operatorname{Spec} R \\ \downarrow & \ulcorner & \downarrow r \\ \operatorname{Spec} R' & \longrightarrow & \mathcal{M}_{\mathrm{FG}} \end{array}$$

The answer is that you get E_B.

Proof. We'll first prove it in a special case. $R = R' = L$, then $E_R \simeq E_{R'} \simeq MP$. We get $MP_0 MP = \big((MU_* MU)[\beta^{\pm 1}, (\beta')^{\pm 1}]\big) \cong (\pi_* MU)[b_1, b_2, \dots][\beta^{\pm 1}, (\beta')^{\pm 1}]$ which is the correct B. [[★★★ did I miss something?]]

Now let's look more generally. Assume that our FGLs over R and R' have coordinates. Then we get $(E_R \otimes E_{R'})_0(X) \cong (E_R)_0(E_{R'} \otimes X) \cong R \otimes_L (MP_0(E_{R'} \otimes X)) \cong \dots R \otimes_L (MP \otimes MP)_0(X) \otimes_L R' = R \otimes_L p^* \mathcal{F}_X \otimes_L R'$. The conclusion is that

$$(E_R \otimes E_{R'})_0(X) \cong R \otimes_L p^* \mathcal{F}_X \otimes_L R'$$

[[★★★ Now there is a larger diagram which extends the previous case. It has $\operatorname{Spec} B$ in the upper left corner, and what we have show is that $B \cong R \otimes_L MP_0 MP \otimes_L R'$]]. □

Corollary 10.7. *Suppose that* $q : \operatorname{Spec} R \to \mathcal{M}_{\mathrm{FG}}$ *is flat. Then* E_R *has a canonical ring structure.*

Proof. Why is this? well we look at the diagram for $E_R \otimes E_R = E_B$, where $\operatorname{Spec} B \to \mathcal{M}_{\mathrm{FG}}$ is the pull-back of $\operatorname{Spec} R$ and $\operatorname{Spec} R$ over $\mathcal{M}_{\mathrm{FG}}$. There is the diagonal map $\operatorname{Spec} R \to \operatorname{Spec} B$, which gives rise to maps,

$$E_R \otimes E_R \cong E_B \to E_R$$

which is commutative and associative up to homotopy. □

Exercise 10.1. $\mathcal{F}_{E_R} \cong (q_* R)$

11 Morava Stabilizer Groups

The moduli stack f formal gorups has a stratification and a typical stratum looks like $\operatorname{Spec} \overline{\mathbb{F}}_p/\mathbb{G}$ where $\mathbb{G}$ is a certain discrete group. We would like to understand this group.

Recall $f(x, y) \in \overline{\mathbb{F}}_p[[x, y]]$ a formal group law of height $n < \infty$. We have a ring $\mathcal{E}nd(f)$, which is a division algebra $D = \mathcal{E}nd(f)[p^{-1}]$ after inverting the prime p. The center of D is isomorphic to $\mathbb{Q}_p$. The dimension of D over $\mathbb{Q}_p$ is n^2.

We have a short exact sequence,

$$0 \to \mathcal{E}nd(f)^\times \to Aut(\overline{\mathbb{F}}_p, f) \to Gal(\overline{\mathbb{F}}_p/\mathbb{F}_p) \to 0$$

The middle group is the *Morava Stabilizer group*.

we have another short exact sequence,

$$0 \to \mathcal{E}nd(f)^\times \to D^\times \xrightarrow{v} \mathbb{Z} \to 0$$

which maps to the previous sequence. Here v is a valuation.

$G = \operatorname{Spf} \overline{\mathbb{F}}_p[[z]]$ lives over $\operatorname{Spec} \overline{F}_p$ and we have a diagram

$$\begin{array}{ccccc} G & \xrightarrow{\text{rel frob}} & G^{(p)} & \longrightarrow & G \\ \downarrow & & \downarrow & \ulcorner & \downarrow \\ \operatorname{Spec} \overline{\mathbb{F}}_p & \xrightarrow[id]{} & \operatorname{Spec} \overline{\mathbb{F}}_p & \xrightarrow[\text{Frob}]{} & \operatorname{Spec} \overline{\mathbb{F}}_p \end{array}$$

$x \in \mathcal{E}nd(f)$ then the valuation of x is n if $x(t) = ct^{p^n} + \cdots$. We can associate to x an automorphism. x itself is not an automorphism, but we have a factorization,

$$x : G \overset{(\text{rel frob})^n}{\to} G^{(p^n)} \overset{\cong}{\to} G$$

On the ground field we are taking the n^{th} power of the frobenius.

Recall that $Gal(\overline{\mathbb{F}}_p/\mathbb{F}_p) \cong \hat{\mathbb{Z}}$ is the completion of $\mathbb{Z}$ (it is a profinite group). So the Morava Stabilizer group is almost $D^\times$.

Recall that $[D] \in Br(\mathbb{Q}_p) \cong \mathbb{Q}/\mathbb{Z}$. For just a moment let D be any division algebra with center $\mathbb{Q}_p$. Then there exists a valuation $v : D^\times \to \mathbb{Z}$. Valuation $\mathcal{O} = \{x \in D \mid v(x) \geq 0\}$. $\mathfrak{m} = \{x \in D \mid v(x) > 0\}$. We have a finite field $\mathcal{O}/\mathfrak{m}$. Then $D^\times$ acts on $\mathcal{O}/\mathfrak{m}$ by conjugation and $\mathcal{O}^\times$ acts trivially. So we get an action of $D^\times/\mathcal{O}^\times \cong \mathbb{Z}$ (this isomorphism is induced by the valuation v). This surjects onto the group $Gal(\mathcal{O}/\mathfrak{m}/\mathbb{F}_p)$. [[★★★ Why is this map surjective?]]

This latter group contains the element given by the Frobenius, which is invertible. This means that this map factors through $\mathbb{Z}/v(p)$.

So we may choose an element such that conjugation by x is the Frobenius. Then and $[D] \mapsto v(x)/v(p)$ is well-defined in $\mathbb{Q}/\mathbb{Z}$.

Claim 11.1. Our D has invariant $\frac{1}{n}$, i.e. since $v(p) = n$, any element x of valuation 1 induces the Frobenius via conjugation. ⋄

$Gal(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ has an outer action on $\mathcal{E}nd(f)^\times$. There is a quotient $\mathcal{E}nd(f)^\times \to \overline{\mathbb{F}}_p^\times$, and the outer action descends to an actual action on this quotient. This is the obvious action.

Remark 11.2. We have ${\mathcal{M}_{\mathrm{FG}}}^{(n)} \to \operatorname{Spec} \mathbb{F}_p$, and the above exact sequences come from the étale fundamental group exact sequences... [[★★★ Eh?]] ◇

Earlier we saw that there was a formal group law of height n over $\mathbb{F}_p$. We know there is a unique one over $\overline{\mathbb{F}}_p$. [[★★★ ???]] So to get a FGL of height n over....

FGLs of height n over $\mathbb{F}_p$ up to isomorphism correspond bijectively elements of $D^\times$ of valuation 1 up to conjugation. A generalization of this shows that FGLs of height n over $\mathbb{F}_{p^k}$ up to isomorphism correspond bijectively elements of $D^\times$ of valuation k up to conjugation. When $k = n$ there is a canonical such element and so there is a *canonical* FGL over $\mathbb{F}_{p^n}$ of height n. It is characterized by $[p](t) = t^{p^n}$. All endomorphisms of this FGL f are defined over $\mathbb{F}_{p^n}$.

What this means is that we didn't have to go to $\overline{\mathbb{F}}_p$. We could have done the same thing with $\mathbb{F}_{p^n}$. We get an exact sequence,

$$0 \to \mathcal{E}nd(f)^\times \to Aut(\mathbb{F}_{p^n}, f) \to Gal(\mathbb{F}_{p^n}/\mathbb{F}_p) = \mathbb{Z}/n \to 0$$

Now the middle guy is isomorphic to $D^\times/p^{\mathbb{Z}}$. This is sometimes also called the Morava stabilizer group.

12 What we have done and where we are going...

Let's say our goal is to compute $\pi_*\mathbb{S}$. One approach to computing this is via the Adams-Novikov [[★★★ sp?]] spectral sequence.

$$H^*(\mathcal{M}_{\mathrm{FG}}, \omega^k) \Rightarrow \pi_*\mathbb{S}$$

The spectral sequence starts with the cohomology of the algebraic stack. This is a complicated algebraic stack and we have been trying to understand it through a filtration.

By analogy, let's replace $\mathcal{M}_{\mathrm{FG}}$ by $\operatorname{Spec} \mathbb{C}[x] = \mathbb{A}^1$, and there is a point $pt \in \mathbb{A}^1$ $(x = 0)$. This is a stratification of $\mathbb{A}^1$. How can we understand modules over $\mathbb{A}^1$ just in terms of the stratification of $\mathbb{A}^1$?

$$0 \to M \to M[x^{-1}] \to M[x^{-1}]/M \to 0$$

Now let's say we have an algerbraic variety Y and $D \subset Y$ a divisor and $U = Y \smallsetminus D$. Furthermore suppose that we understand D and U. For any sheaf $\mathcal{F}$ on Y, we have,

$$0 \to \mathcal{F} \to j_*j^*\mathcal{F} \to j_*j^*\mathcal{F}/\mathcal{F} \to 0$$

(the first map is an injection if $\mathcal{F}$ is sufficiently nice (flat?)).

What we are really after is something much more complicated. First it is a stack. Also there is more than one strata. What we get, instead of an exact sequence, is the Cousin Spectral Sequence which goes from the cohomology of q-coherent sheaves on $\mathcal{M}_{\mathrm{FG}}$, set-theoretically supported on the strata ${\mathcal{M}_{\mathrm{FG}}}^n$. There is a convergence issue. This comes from the question of whether it is okay to ignore the locus where the height is infinite?

Now let's go back to the first example. The module $M[x^{-1}]/M$ is not scheme-theoretically supported at the point, just 'set-theoretically'. However it has a filtration,

$$x^{-1}M/M \supseteq x^{-2}M/M \supseteq \dots$$

by modules where the successive quotients are scheme-theoretically supported on the point. More generally, colim $\mathcal{F}(nD)/\mathcal{F} \cong j_* j^* \mathcal{F}$. This ultimately gives us another spectral sequence whose input is the cohomology of sheaves on ${\mathcal{M}_{\mathrm{FG}}}^{(n)}$. These are n-Bockstein spectral sequences. This is essentially the same calculation as the cohomology of a profinite group (the Morava Stabilizer group).

So if you want even a little bit of an answer, there is a lot of computation of spectral sequences. The first spectral sequence (ANSS) is one which goes from algebra to topology. The rest are algerbraic tricks or mechanisms which reduce it to easier algebraic problems. The rest of this course will be trying to commute the topology earlier and earlier into this sequence of calculations. Can we use this as an organizing principle for algebraic topology.

13 Categorical Background

Recall, we hgave a functor $\mathsf{Spectra} \to QCoh(\mathcal{M}_{\mathrm{FG}})$ which sends a spectrum X to $MP_*(X)$ as MP_*MP-comodule. Our goal is to study spectra using the geometry of $\mathcal{M}_{\mathrm{FG}}$.

Let X be an algebraic variety. What does it mean to talk about the topology/geometry of the variety? You have Zariski closed subsets $Y \subset X$. If you have such a Zariski closed subset, then you have a sub-category of those sheaves which are supported on Y. This such category is stable under all colimits.

Suppose we are given a subcategory $\mathcal{C} \subset \mathsf{Spectra}$ which is closed under shifts and colimits, and (technical condition) there exists a small category $\mathcal{C}_0$ which generatees $\mathcal{C}$ under homotopy colimits (for example if $\mathcal{C} =$ all spectra, we can take $\mathcal{C}_0$ to be finite spectra). Then the inclusion $F : \mathcal{C} \to \mathsf{Spectra}$ preserves hocolims, and so by the adjoint functor theorem (which needs this technical condition) F has a right adjoint G. This induces a cofiber sequence,

$$G(X) \to X \to L(X)$$

We have for all $Y \in \mathcal{C}$, $G(X)^Y \simeq X^Y \to L(X)^Y$, implies that $L(X)^Y \simeq 0$, i.e. any map from Y to $L(X)$ is null-homotopic. We define $\mathcal{C}^\perp$ to be all Z such that for all $Y \in \mathcal{C}$, every map $f : Y \to Z$ is null-homotopic. Then $X \to L(X)$ is universal for all maps $X \to Z$ with $Z \in \mathcal{C}^\perp$. Thus we can think of L as the *left-adjoint* to the inclusion $\mathcal{C}^\perp \subset \mathsf{Spectra}$

Example 13.1 (Bousfield Localization)**.** Let E be a spectrum, and let $\mathcal{C}_E = \{X \mid E_*(X) \cong 0\}$, i.e. $E \otimes X \simeq 0$. We say that X is E-acyclic if it belongs to $\mathcal{C}_E$. Moreover $\mathcal{C}_E$ satisfies the technical condition (by an argument of Bousfield). Thus there exists a right-adjoint $G_E : \mathsf{Spectra} \to \mathcal{C}_E$ and a left-adjoint $L : \mathsf{Spectra} \to \mathcal{C}_E^\perp$.

Definition 13.2. A spectrum X is *E-local* if $X \in \mathcal{C} + E^\perp$. ◇

Then we have $G_E(X) \to X \overset{\alpha}{\to} L_E(X)$ such that $L_E(X)$ is E-local and $E_*(X) \to E_*(L_E(X))$ is an isomorphism. ◇

Remark 13.3. Suppose that E is a ring spectrum. Then any E-module spectrum X is E-local. Why? Let Y be E-acyclic, then for all maps $f : Y \to F$, the composite

$$Y \to X \to E \otimes X \to X$$

is just f by the module property. On the other hand, it factors as Factors $Y \to E \otimes Y \to E \otimes X \to X$, which is zero since Y is E-acyclic. ◇

Now E is A_∞, then we get

$$X \to (X \otimes E \rightrightarrows X \otimes E \otimes E \Rrightarrow \cdots)$$

Then we get a map $X \to L_E X \xrightarrow{\beta} Tot(X \otimes E^{*+1})$. In many good cases (which ones depend on the conditions for convergence of the ASS) this β is an equivalence.

Remark 13.4. All this works much more generally. For example we can replace $\mathsf{Spectra}$ by chain complexes of abelian groups, i.e. use module spectra over $H\mathbb{Z}$. ⋄

Example 13.5. Let $E = \mathbb{Q}$, then a complex A_* is E-local if and only if $H_*(A_*)$ is a $\mathbb{Q}$-vector space. The cofiber sequence becomes,

$$G_{\mathbb{Q}}(A_*) \to A_* \to A \otimes \mathbb{Q} = L_E(A_*).$$

⋄

Example 13.6. Let $E = \mathbb{Z}/p$. Then A_* is E-acyclic if and only if $A_* \otimes^{\mathbb{L}} \mathbb{Z}_p$ is acyclic, i.e.

$$A_* \xrightarrow{p} A_* \to A_* \otimes^{\mathbb{L}} \mathbb{Z}/p$$

is an isomorphism, i.e. $H_*(A_*)$ is a $\mathbb{Z}[\frac{1}{p}]$-module. ⋄

Note: For all A_*, $A_* \otimes^{\mathbb{L}} \mathbb{Z}_p$ is E-local, as is $A_* \otimes^{\mathbb{L}} \mathbb{Z}/p^n$. Since the E-local objects are also invariant under homotopy limits, we have

$$\hat{A}_* \cong \operatorname{holim}(A_* \otimes^{\mathbb{L}} \mathbb{Z}/p^n)$$

is E-local. Then $A_* \to \hat{A}_*$ exhibits $\hat{A}_*$ as an E-localization of A_*. In other words, $L_E : D(\mathsf{Ab}) \to D(\mathsf{Ab})$ is the left-derived functor of p-adic completion.

Vague statements: Bousfield localiation can sometimes behave like algebraic localization. Sometimes the behaviour is different (sometimes like algebriac completions). In general it feels like a mixture of this two.

Goal: Describe the first case axiomatically.

Now let's go back to the first situation: $\mathcal{C} \subset \mathsf{Spectra}$.

Theorem 13.7. *The following are equivalent*

1. $\mathcal{C}^\perp$ *is stable under hocolims*
2. L *preserves hocolims*
3. G *preserves hocolims*
4. *for all* X, $L(X) \simeq K \otimes X$ *for some* K

Proof. (2) ⇒ (1): $X \in \mathcal{C}^\perp$ iff $X \simeq L(X)$, and $X \simeq \operatorname{hocolim} X_\alpha$, and so $LX \simeq L \operatorname{hocolim} X_\alpha \simeq \operatorname{hocolim} LX_\alpha$.

(1) ⇒ (2): $X = \operatorname{hocolim} X_\alpha$

$$\operatorname{hocolim} G(X_\alpha) \to \operatorname{hocolim} X_\alpha \to \operatorname{hocolim} LX_\alpha$$

(2) ⇔ (3) $G \to id \to L$... ?

(2) ⇔(4) follows from $F : \mathsf{Spectra} \to \mathsf{Spectra}$ which preserves hocolim, then $X \otimes F(\mathbb{S}) \to F(X)$ implies, $F(X) \simeq K \otimes X$. □

[[★★★ clean up the above proof]]
Idea: smashing localizations $\Leftrightarrow$ 'restricting to an open'.
note: If L is a smashing localization, then $\mathcal{C}$ consists of the $L(\mathbb{S})$-acyclics.

Example 13.8. If $\mathcal{C}$ is generated under hocolim by $\mathcal{C}_0 \subseteq \mathcal{C}$ where $\mathcal{C}_0$ consists of finite spectra. Then L is smashing. We can check (1). We have $X \in \mathcal{C}^{\perp}$ if and only if $X^Y \simeq 0$ if $Y \in \mathcal{C}$ is equivalent to $X^Y \simeq 0$ if $Y \in \mathcal{C}_0$. Since each such Y is finite, the class of such X is preserved under homotopy colimits. $\diamond$

Question 13.9. Are there other smashing localizations? (this implies the telescope conjecture. (believed to be false)).

14 Lubin-Tate Theory

k perfect field of characteristic p. $f \in k[[x, y]]$ FGL of height n.

Definition 14.1. A infinitesimal thikening of k is a local ring A, $A/\mathfrak{m}_A \cong k$, where $\mathfrak{m}_A^n = 0$ for large enough n, and $\mathfrak{m}_A^k/\mathfrak{m}_A^{k+1}$ is a finite dimensional vector space. i.e. A is a local Artin ring with residue k. $\diamond$

Definition 14.2. $Def(A)$ is FGL over A which map to $f \in FGL(k)$, up to isomorphisms which restrict to the identity over k. $\diamond$

claim: If $\tilde{f}$ is a deformation of f over A, then $\tilde{f}$ has no automorphisms reducing to the identify of f. Hence it is okay to view $Def(A)$ as a set (rather then, say, a groupoid).

Proof. Let $g = g(t) \in A[[t]]$ be such an automorphism of $\tilde{f}$. We will prove that $g(t) \equiv t$ modulo $\mathfrak{m}_A^n$ using induction on n. It is true for $n = 1$ by assumption. This implies that $g(t) = t$.

So suppose that $g(t) \equiv t$ module $\mathfrak{m}_A^n$. Let B be the A-algebra classifying automorphisms of $\tilde{f}$. I.e. $\mathrm{Hom}_A(B, A')$ is automorphisms of $\tilde{f}$ over A'. We have two automorphisms over A, the identity and $g(t)$. Thus we get,

$$B \rightrightarrows A \to A/\mathfrak{m}_A^{n+1} \to A/\mathfrak{m}_A^n$$

We know they agree after composing to $A/\mathfrak{m}_A^n$. We want to show they agree at $A/\mathfrak{m}_A^{n+1}$. Call these maps λ and λ'. We know that the difference $\lambda - \lambda'$ is a derivation of B into $V = \mathfrak{m}_A^n/\mathfrak{m}_A^{n+1}$. Morevoer since these are A-algebra homomorphisms, we have that $\lambda - \lambda'$ is A-linear.

Let $d = \lambda - \lambda'$. The $d : B \to V$ is A-linear. d factors as $B \to B/\mathfrak{m}_A B \to V$, and this middle term is $B \otimes_A k$ which represents an automorphisms of f of height exactly n. This is also a direct limit of finite étale extensions of k. [[★★★ Now the previous arguments show that there aren't any so $d = 0$.???]] $\square$

So we've seen that $Def(A)$ is a set. There is a gluing property, if $A \to B$ and $C \to B$ are surjective, then the square

$$\begin{array}{ccc} Def(A \times_B C) & \to & Def(A) \\ \downarrow & \ulcorner & \downarrow \\ Def(C) & \longrightarrow & Def(B) \end{array}$$

is a pull-back diagram. This says that to give a FGL over $\mathrm{Spec}(A) \cup_{\mathrm{Spec}(B)} \mathrm{Spec}(C)$ we just need to specify it over each open set and their (closed!) intersection. (at least for deforming rings like A, B, C.)

goal: Understand the functor $A \mapsto Def(A)$ as a map $\mathcal{S}pec\, A \to \mathcal{M}_{\mathrm{FG}}$ extending the map $\mathrm{Spec}\, k \to \mathcal{M}_{\mathrm{FG}}$ given by f (note $\mathrm{Spec}\, k \to \mathrm{Spec}\, A$).

Claim 14.3. $Def(A) \cong \mathrm{Hom}_k(W(k)[[v_1, \dots, v_{n-1}]], A)$ is an isomorphism of functors. ⋄

Here $W(k)$ is a complete DVR with uniformizer p, such that $W(k)/p \cong k$. [[★★★ Something about Witt vectors.]] The ring $W(k)[[v_1, \dots, v_{n-1}]]$ is called the Lubin-Tate ring. Let's call it R for notational simplicity.

Proof. We can lift the map from the Lazard ring to k to a map $\phi : L_{(p)} = \mathbb{Z}_{(p)}[t_1, t_2, \dots] \to R$, where $L_{(p)}$ is the localized Lazard ring. We can choose ϕ such that $\phi(v_i) = v_i$ for $0 < i < n$, and can lift the rest arbitrarily.

Strategy of proof. We use two properties of $Def(A)$ and $\mathrm{Hom}_k(R, A)$. The first property is the gluing property we saw earlier. Both of these functors satisfy this (note that after we identify them we see that they both satisfy gluing for arbitrary fiber products! (because $\mathrm{Hom}_k(R, -)$ satisfies this.)). The second property is "formal smoothness": If $A \to B$ is surjective, then $Def(A) \to Def(B)$ is also surjective.

Suppose that we want to understand $Def(A)$, where A is an infinitesimal thickening of k. It is a fintite Artin local ring and hence has a length. We will induct on the length $\ell(A)$. If $A = k$, then $Def(A) = pt$. More generally, choose $x \in A$, non-zero, such that $m_A \cdot x = 0$. Consider the map $A \to A/x$. There is a map,

$$k[x]/(x^2) \times_k k[y]/y^2 \to k[z]/(z^2)$$

given by $x \mapsto z$, and $y \mapsto z$. By the gluing formula, this induces a multiplication

$$Def(k[x]/(x^2)) \times Def(k[x]/(x^2)) \to Def(k[x]/(x^2)).$$

This is a group structure on $Def(k[x]/(x^2))$.

There is also a map $k[x]/(x^2) \times_k A \to A$ (careful this is not a tensor product, it is the fiber product $A[x]/(x^2, m_A \cdot x)$). Then we get a fiber product square,

$$\begin{array}{ccc} k[x]/(x^2) \times_k A & \longrightarrow & A \\ \downarrow & \ulcorner & \downarrow \\ A & \longrightarrow & A/x \end{array}$$

So this means we get another pull-back square,

$$\begin{array}{ccc} V \times Def(A) & \rightarrow & Def(A) \\ \downarrow & \ulcorner & \downarrow \\ Def(A) & \rightarrow & Def(A/x) \end{array}$$

Thus V acts freely and we have $Def(A)/V \cong Def(A/x)$.

Similarly, $\mathrm{Hom}(R, k[x]/(x^2)) = W$, and W acts freely on $\mathrm{Hom}(R, A)$ with quotient $\mathrm{Hom}(R, A/x)$.

Suppose that we knew that the map $\mathrm{Hom}(R, A/x) \to Def(A/x)$ is an isomorphism, then $\mathrm{Hom}(R, A) \to Def(A)$ is an isomorphism provided that $W \to V$ is an isomorphism.

Now $\mathrm{Hom}(R, k[x]/x^2) \to Def(k[x]/x^2) \to k^{n-1}$. What is this second map? $[p]_{\tilde{f}}(t) = \sum c_i t^i$ and we get $\tilde{f} \mapsto (c_p, c_{p^2}, \ldots, c_{p^{n-1}})$. Similarly, $\mathrm{Hom}(R, k[x]/x^2)$ is precisely k^{n-1}. So the induced map is one like: $k^{n-1} \to k^{n-1}$. The kernel consists of those deformations which do not change the height. But the map $Def(k[x]/x^2) \to k^{n-1}$ is injective [[★★★ Why?]].

$\square$

Next time we will look at the cohomology theory we get from R...

15 Morava K-Theories

$E(n)$ is variably called Morava E-theory or Lubin-Tate Theory.

[late to class]

X is $E(n)$-acyclic is equivalent to the sheaf $\mathcal{F}_{\Sigma^k X}$ is supported on $\mathcal{M}_{\mathrm{FG}}{}^{\geq n+1}$. Associated to $E(n)$ is a localization $L_{E(n)}$ which 'feels like' restriction to $\mathcal{M}_{\mathrm{FG}}{}^{\leq n}$

Theorem 15.1 (Smash Product Theorem)**.** *$L_{E(n)}$ is smashing, i.e. commutes with colimits.*

Observation: $R \cong W(k)[[v_1, \dots, v_{n-1}]]$ depends functorially on the initial data: a field k and a formal group of height n over k. Note that if A is any complete local noetherian ring with residue field k, then

$$\mathrm{Hom}_k(R, A) \to Def(A) \cong \lim Def(A/\mathfrak{m}_A^n).$$

We learn that this first map is an isomorphism. This means that R is functorially determined by the data we started with.

Remark 15.2. $k = \mathbb{F}_p$, $Aut(k; f)$ is the Morava stabilizer group G. Therefore G acts on $W(\overline{\mathbb{F}}_p)[[v_1, \dots, v_{n-1}]]$, and therefore G acts on $E(n)$ in the homotopy category of spectra. (the action is complicated, it is not linear). This can be improved. Using the Hopkins-Miller theorem, you can see that $E(n)$ is an E_∞-ring spectrum and this is an ∞-action. ◇

$$\mathcal{M}_{\mathrm{FG}}{}^{n} \subseteq \mathcal{M}_{\mathrm{FG}}{}^{\leq n} \subseteq \mathcal{M}_{\mathrm{FG}} \times \mathrm{Spec}(\mathbb{Z}_{(p)})$$

$E(n)$ is associated to $\mathcal{M}_{\mathrm{FG}}{}^{\leq n}$ which is an open substack (hence flat) and so we can use Landweber's theorem. This won't work for $\mathcal{M}_{\mathrm{FG}}{}^{n}$ since this is a *closed* substack. Our goal is to nevertheless construct Morava K-theory $K(n)$ which 'knows about' $\mathcal{M}_{\mathrm{FG}}{}^{n}$.

$$\pi_* MU_{(p)} \cong L_{(p)} \cong \mathbb{Z}_{(p)}[t_1, t_2, \dots]$$

Since p is invertible, we can assume $v_1 = t_{p^i - 1}$ for $i > 0$. Be convention $t_0 = v_0 = p$.

Fact: MU exists as a ring object in a robust category of ring spectra. (eg. symetric spectra). It is an E_∞-ring spectrum. There is a theory of modules over $MU_{(p)}$ which have a *relative smash product* $(M, N) \mapsto M \otimes_{MU_{(p)}} N$ which is nicely behaved, like usual tensor products (commutes with colimits in both variables).

$M(k)$ is defined to be the cofiber of $t_k : \Sigma^{2k} MU_{(p)} \to MU_{(p)}$.

Claim 15.3. $M(k)$ has the structure of a unital homotopy associative algebra over $MU_{(p)}$. ◇

The unit is given by $MU_{(p)} \to M(k)$. We see that $\pi_* M(k) = L_{(p)}/t_k$. What is $M(k) \otimes_{MU_{(p)}} M(k)$? It has maps to $M(k)$.

$$\begin{array}{ccc} \Sigma^{4k} MU_{(p)} & \xrightarrow{t_k} & \Sigma^{2k} MU_{(p)} \\ {\scriptstyle t_k}\downarrow & & \downarrow{\scriptstyle t_k} \\ \Sigma^{2k} MU_{(p)} & \xrightarrow[t_k]{} & MU_{(p)} \end{array}$$

$M(k)$ is the total cofiber of the above diagram. Define X to be the total cofiber of the diagram

$$\begin{array}{ccc} 0 & \xrightarrow{t_k} & \Sigma^{2k} MU_{(p)} \\ {\scriptstyle t_k}\downarrow & & \downarrow{\scriptstyle t_k} \\ \Sigma^{2k} MU_{(p)} & \underset{t_k}{\to} & MU_{(p)} \end{array}$$

i.e. the cofiber of

$$\Sigma^{2k} MU_{(p)} \oplus \Sigma^{2k} MU_{(p)} \to MU_{(p)}$$

Then we have a partially defined multiplication, $M(k) \otimes_{MU_{(p)}} M(k) \leftarrow X \to M(k)$. Let Y be the total cofiber of

$$\begin{array}{ccc} \Sigma^{4k} MU_{(p)} & \xrightarrow{t_k} & 0 \\ {\scriptstyle t_k}\downarrow & & \downarrow{\scriptstyle t_k} \\ 0 & \underset{t_k}{\longrightarrow} & 0 \end{array}$$

have a cofiber sequence,

$$\Sigma^{-1} Y \to X \to M(k) \otimes_{MU_{(p)}} M(k)$$

So the obstruction to extending the multiplication is $\pi_{4k+1} M(k)$, which vanishes.

So we get a multiplication $m : M(k) \otimes_{MU_{(p)}} M(k) \to M(k)$

Claim 15.4. m is associative. i.e.

$$f - g : M(k) \otimes_{MU_{(p)}} M(k) \otimes_{MU_{(p)}} M(k) \to M(k)$$

is null-homotopic. ◇

Now there is a cubical diagram, and we make a similar argument to the previous one. $M(k) \otimes_{MU_{(p)}} M(k) \otimes_{MU_{(p)}} M(k)$ can be identified with the total colimit of this diagram. We do a similar trick by erasing the upper corner, and taking the cofiber of that diagram. Call it Y. Then $f - g|_Y$ is null-homotopic. A similar cofiber sequence gives another obstruction to extending which is again an odd class, hence vanishes.

In fact a similar construction shows that for all the associahedra the relevant obstruction vanishes and so we get an A_∞ multiplication.

Definition 15.5. $K(n)$ is the smash product (over $MU_{(p)}$) of $MU_{(p)}[t_{p^n-1}^{-1}]$ with $M(k)$ for all $k \neq p^n - 1$. ◇

We get a spectrum $K(n)$ with $\pi_* K(n) \cong \mathbb{F}_p[v_n^{\pm 1}]$, which is homotopy associative multiplication. It is complex oriented $MU \to K(n)$.

Warning:

How unique is the multiplication on the $M(k)$? (the obstructions vanish, but there is a choice of a trivialization of such obstructions). The possible multiplications up to homotopy form a torsor P for $\pi_{4k+2}M(k) \neq 0$. P consists of certain maps $M(k)^{\otimes 2} \to M(k)$. To make this commutative, we need to trivialize the obvious $\mathbb{Z}/2$-action. We get $H^1(\Sigma_2; \pi_{4k+2}M(k))$, which is zero if the prime $p \neq 2$.

It turns out that we are okay if $p \neq 2$. If $p = 2$ there is no E_∞-structure on $K(n)$. However in either case, we are still okay, because we just needed to know that the cohomology of $\mathbb{CP}^\infty$ is a commutative ring. This is fine since we have,

$$K(n)^*(\mathbb{CP}^\infty) \cong MU^*(\mathbb{CP}^\infty) \otimes_L \mathbb{F}_p[v_n^{\pm 1}]$$

Question: How unique is $K(n)$? (Ans: Somewhat unique...)

16 Bousfield Classes

Recall that $\pi_* MU_{(p)} \cong L_{(p)} \cong \mathbb{Z}_{(p)}[t_1, t_2, \dots]$, that $M(k)$ is the cofiber of

$$t_k : \Sigma^{2k} MU_{(p)} \to MU_{(p)}$$

and that $K(n)$ is the smash product over $MU_{(p)}$ of $MU_{(p)}[v_n^{-1}]$ will all $M(k)$ for $k \neq p^n - 1$, where $v_i = t_{p^i - 1}$, and $v_0 = t_0 = p$. Then we have

$$\pi_* K(n) \cong \pi_* MU_{(p)}[v_n^{-1}]/(t_0, t_1, \dots) \cong \mathbb{F}_p[v_n^{\pm 1}].$$

There are some choices that went into constructing this as a ring spectrum and even as a spectrum. We will see that as a spectrum (but not as a ring spectrum) it is independent of these choices.

Consequently we have a FGL of height n over $\mathbb{F}_p[v_n^{\pm 1}]$. There are several FGLs and a multiplication on $K(n)$ will give you a formal group (and hence a formal group law). [[★★★ There is actually an equivalence here...]]

Definition 16.1. Two spectra E and E' are *Bousfield Equivalent* if for all X, X is E-acyclic is equivalent to X is E'-acyclic. ◇

Example 16.2. Let E be a complex oriented p-local spectrum which is Landweber exact. $R = \pi_{even} E$ and $\phi : \operatorname{Spec} R \to \mathcal{M}_{\mathrm{FG}}$ is flat. If the image of ϕ is $\mathcal{M}_{\mathrm{FG}}{}^{\leq n}$, then $E \sim E(n)$. ◇

Proof. X is E-acyclic iff for all n $0 \cong E_* X \cong MU_*(X) \otimes_L \pi_* E \cong \phi^* \mathcal{F}_X$ which is equivalent to $\mathcal{F}_X|_{\mathcal{M}_{\mathrm{FG}}{}^{\leq n}} = 0$. □

Theorem 16.3. *$E(n) \sim E(n-1) \times K(n)$, i.e. for any spectrum X, X is $E(n)$-acyclic if and only if X is $E(n-1)$-acyclic and X is $K(n)$-acyclic.*

For $0 \leq m \leq n$, let $Z(m) =$ the smash product (over $MU_{(p)}$-modules) of $MU_{(p)}[v_n^{-1}]$ with $M(k)$ for $k \neq p^a - 1$. Then $\pi_* Z(m) \cong \mathbb{Z}_{(p)}[v_1, v_2, v_3, \dots, v_n^{\pm 1}]/(v_0, v_1, \dots, v_{m-1})$. Observe that $Z(n) = K(n)$, and $Z(0)$ is Landweber exact,

$$\operatorname{Spec} \mathbb{Z}_{(p)}[v_1, \dots, v_n^{\pm}] \to \mathcal{M}_{\mathrm{FG}}$$

has image $\mathcal{M}_{\mathrm{FG}}{}^{\leq n}$, so $Z(0) \sim E(n)$.

Easy direction: If X is $E(n)$-acyclic, then X is $E(n-1)$-acyclic and is $K(n)$-acyclic [[★★★ why?]]. If X is $E(n)$-acyclic, then this is equivalent to X being $Z(0)$-acyclic, i.e. $X \otimes Z(0) \simeq 0$. Thus we have,

$$X \otimes K(n) \simeq X \otimes Z(n) \simeq (X \otimes Z(0)) \otimes_{MU_{(p)}} (\text{smash prdt of } M(p^a - 1)) \simeq 0$$

so X is $K(n)$-acyclic.

Now suppose that X is $K(n)$- and $E(n)$-acyclic. Then it is also $Z(0)$-acyclic. We will show that X is $Z(i)$-acyclic for each $0 \leq i \leq n$. We will prove this by descending on i. It is true, by assumption for $i = n$. Assume that $0 \leq i < n$ and X is $Z(i+1)$-acyclic. We want to show it is $Z(i)$-acyclic. We have a cofiber sequence,

$$\Sigma^{2(p^i-1)} Z(i) \xrightarrow{v_i} Z(i) \to Z(i+1).$$

Thus, since X is $Z(i+1)$-acyclic, we know that v_i is invertible on $X \otimes Z(i)$. Thus $X \otimes Z(i) \simeq X \otimes Z(i)[v_i^{\pm 1}]$.

Now it suffices to show that X is $Z(i)[v_i^{\pm 1}]$-acyclic. We have,

$$Z(i)[v_i^{\pm 1}] \simeq Z(0)[v_i^{-1}] \otimes_{MU_{(p)}} (\text{some } M(p^j - 1))$$

where $0 \leq j < i$. So it suffice to prove X is $Z(0)[v_i^{-1}]$-acyclic. This is a Landweber exact cohomology theory with

$$\pi_* Z(0)[v_i^{-1}] \cong \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_{i-1}, v_i^{\pm 1}, v_{i+1}, \ldots, v_n^{\pm 1}].$$

The image of $\operatorname{Spec} R \to \mathcal{M}_{\text{FG}}$ in this case is $\mathcal{M}_{\text{FG}}{}^{\leq i}$. Hence $Z(0)[v_i^{-1}] \sim E(i)$, and hence since X is $E(n-1)$-acyclic, it is $E(i)$-acyclic, hence $Z(0)[v_i^{-1}]$-acyclic.

Theorem 16.4 (Hard)**.** *$L_{E(n)}$ is smashing.*

[[★★★ We wont prove this.]] Hence $L_{E(n)} X \simeq X \otimes L_{E(n)}(\mathbb{S})$.

Corollary 16.5. *If X is $E(n-1)$-local, then X is $K(n)$-acyclic.*

Morally, $X \sim \mathcal{F}_X$ is supported on $\mathcal{M}_{\text{FG}}{}^{\leq n-1}$ and hence should vanish on $\mathcal{M}_{\text{FG}}{}^{n}$. But now we can prove this.

Proof. If $X \simeq L_{E(n-1)} X$, then $X \otimes K(n) \simeq L_{E(n)} X \otimes K(n) \simeq L_{E(n-1)} \mathbb{S} \otimes X \otimes K(n) \simeq X \otimes L_{E(n-1)} K(n)$. Hence we want $L_{E_(n-1)} K(n) \simeq 0$. This is equivalent to $K(n) \cong E(n-1)$. This is a complex orientable cohomology theory whose FGL has bot height exactly n and height strictly less than n. This is impossible, unless it is the zero cohomology theory. □

Suppose that M is a finitely generated abelian group. Suppose we want $M_{(p)}$. We can do two things, $\hat{M} = \lim M/p^k M$, and take $M \otimes \mathbb{Q}$. We have a pullback diagram,

$$\begin{array}{ccc} M_{(p)} & \longrightarrow & \hat{M} \\ \downarrow & \ulcorner & \downarrow \\ M \otimes \mathbb{Q} & & \hat{M} \otimes Q = M \otimes \mathbb{Q}_p \end{array}$$

By analogy, we can consider the pullback,

$$\begin{array}{ccc} X' & \longrightarrow & L_{K(n)}X \\ \downarrow & \ulcorner & \downarrow \\ L_{E(n-1)}X & \longrightarrow & L_{E(n-1)}L_{K(n)}X \end{array}$$

We have $\alpha : X \to X'$.

Claim 16.6. α exhibits X' as an $E(n)$-localization of X. ⋄

[[★★★ I missed the next argument]] X' is $E(n)$-local.

We want that α is an $E(n)$-equivalence. i.e. the fiber of α is $E(n)$-acyclic. This is equivalent to the fiber of α is $K(n)$ is $E(n-1)$-acyclic. i.e. we want the following square to be a homotopy pull-back:

$$\begin{array}{ccc} K(n) \otimes X & \longrightarrow & K(n) \otimes L_{K(n)}X \\ \downarrow & \ulcorner & \downarrow \\ K(n) \otimes L_{E(n-1)}X & \longrightarrow & K(n) \otimes L_{E(n-1)}L_{K(n)}X \end{array}$$

[[★★★ something about the previous lemma]]. We also want the following diagram to commute [[★★★ replace $K(n)$ with $E(n-1)$]]

Upshot: $L_{E(n)}$ is $L_{K(n)} \otimes_{L_{E(n-1)} \circ L_{K(n)}} L_{E(n-1)}$. I.e. $E(n)$-local homotopy theory is determined by $K(n)$-local homotopy theory and $E(n-1)$-local homotopy theory.

Suppose that X is $E(n)$-local. Then it fits in a fiber diagram,

$$\begin{array}{ccc} X & \longrightarrow & L_{K(n)}X \\ \downarrow & \ulcorner & \downarrow \\ L_{E(n-1)}X & \longrightarrow & L_{E(n-1)}L_{K(n)}X \end{array}$$

Conversely, suppose that we have an $K(n)$-local spectrum Y, and as $E(n-1)$-local spectrum Z, and form the pullback square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & \ulcorner & \downarrow \\ Z & \longrightarrow & L_{E(n-1)Y} \end{array}$$

Then $L_{K(n)}X \simeq Y$, and $L_{E(n-1)}X \simeq Z$ [[★★★ missed arguments.]]

So the whole category of $E(n)$-local spectra can be describe as the category of triples $(Y, Z, \alpha : Z \to L_{E(n-1)}Y)$ where Y is $K(n)$-local and Z is $E(n-1)$-local.

17 Is $K(n)$ unique?

Recall, for all p and n we have Morava K-theories $K(n)$. This is a complex oriented cohomology whose formal group has height n. We have $\pi_*K(n) \cong \mathbb{F}_p[v_n^{\pm 1}]$. Recall that this is roughly what we get when we take $MU_{(p)}$ by kill p and all the variables except t_{p^n-1}, and inverting this remaining variable.

Why is $K(n)$ 'nice' or 'good'?

Definition 17.1. An evenly graded commutative ring is a *graded field* if ever non-zero homogeneous element $x \in R_n$ is invertible. Equivalently, either $R = k$ is of degree zero and is a field or $R = k[\beta^{\pm 1}]$, where β is of homogeneous degree $d > 0$. Again k is a field. ◇

If M is a graded R-module then M is free (on homogeneous basis elements). If $R = k$, then this is easy. Let's assume $R = k[\beta^{\pm 1}]$. Then any k-basis of $M_0 \oplus M_1 \oplus \cdots \oplus M_{d-1}$ is an R-basis for M.

Definition 17.2. A ring spectrum E is a *field* if π_*E is a graded field. ◇

Example 17.3. Morava K-theory. ◇

Example 17.4. If k is a field, then Hk is a field. For example $H\mathbb{Q} \simeq K(0)$ and $H\mathbb{F}_p \simeq K(\infty)$. ◇

What we will see is that these are essentially all the fields in spectra. (Any graded field contains a copy of these rings and there is a similar statement for spectra).

Proposition 17.5. *Let E be a field, M an E-module. The M is free, $M \simeq \oplus\Sigma^{k_\alpha}E$.*

Proof. Choose a homogeneous basis for π_*M over π_*E. Then we get $\oplus\Sigma^{k_\alpha}E \to M$ is an isomorphism on π_*. □

Proposition 17.6 (Künneth formula)**.** *E a field, then $E_*(X \otimes Y) \cong E_*(X) \otimes_{\pi_*E} E_*(Y)$.*

Proof.

$$\begin{aligned} E_*(X \otimes Y) &\cong \pi_*(E \otimes X \otimes Y) \\ &\cong \oplus_\alpha \pi_*(\Sigma^{k_\alpha}E \otimes Y) \\ &\cong \oplus E_{*-k_\alpha}(Y) \\ &\cong E_*(X) \otimes_{\pi_*E} E_*(Y). \end{aligned}$$

□

These are essentially the only spectra which have a Künneth formula. Let M and N be $K(n)$-module spectra, π_*M and π_*N are $\mathbb{F}_p[v_n^{\pm 1}]$-modules. If $f : M \to N$ is a map of spectra.

Claim 17.7. $f_* : \pi_*M \to \pi_*N$ is $\mathbb{F}_p[v_n^{\pm 1}]$-linear. ◇

Proof. f is the composition

$$M \overset{\text{unit}}{\to} K(n) \otimes M \overset{id \otimes f}{\to} K(n) \otimes N \overset{\text{action}}{\to} N$$

The middle map and last map induce $\mathbb{F}_p[v_n^{\pm}]$-linear maps. Thus it suffice to consider the case that f is the unit map $M \to K(n) \otimes M$. Since M is free, without loss of generallity we may assume $M = K(n)$, and the map is $u \otimes id : K(n) \to K(n) \otimes K(n)$. There is another map $id \otimes u : K(n) \to K(n) \otimes K(n)$. We want to know that these induce the same map

$$\phi, \psi : \mathbb{F}_p[v_p^{\pm 1}] \to R = \pi_{even} K(n) \otimes K(n).$$

$K(n) \otimes K(n)$ has two complex orientations. Over R we have two formal group laws, with p-series,

$$[p](t) = \phi(v_n)t^{p^n} + \cdots$$
$$[p](t) = \psi(v_n)t^{p^n} + \cdots$$

These formal group laws differ by a coordinate transformation $t \mapsto t + b_1 t^2 + \cdots$. Such a coordinate change cant change the leading coefficient and hence we must have $\psi(v_n) = \phi(v_n)$. □

Corollary 17.8. *Let M be a $K(n)$-module, and let N be a retract of M (in homotopy category of spectra). Then N has a $K(n)$-module structure.*

Proof. To be a retract means that $i : N \to M$ and $r : M \to N$ and $r \circ i = id_N$. Then $i(\pi_* N) \subseteq \pi_* M$, is a direct summand, and is the image of $i \circ r : \pi_* M \to \pi_* M$. By the above claim, this is a map of $\mathbb{F}_p[v_n^{\pm 1}]$-modules, and so $\pi_* N$ is such a module.

Choose a basis for $\pi_* N$ as a $\mathbb{F}_p[v_n^{\pm 1}]$-module. This gives us a map,

$$\oplus \Sigma^{k_\alpha} K(n) \to M \to N$$

which is an isomorphism on homotopy groups and is hence an equivalence. □

Corollary 17.9. *Let E be a field. If $E \otimes K(n)$ is non-zero, then E has a $K(n)$-module structure.*

Proof. E is a retract of $E \otimes K(n)$. □

Claim 17.10. Let E be any complex oriented ring spectrum such that $\pi_* E \cong \mathbb{F}_p[v_n^{\pm 1}]$ with the associated formal group of height n. Then $E \simeq K(n)$. ⋄

Proof. It suffices to show that E is a $K(n)$-module. By the last corollary, this is true if $E \otimes K(n)$ is non-zero. So suppose, for contradiction, that $E \otimes K(n) \simeq 0$. Then by height [[★★★ ???]] $E \otimes E(n-1) \simeq 0$. This implies that $E \otimes E(n) \simeq 0$. But we have

$$\pi_0(E \otimes E(n))(MU_{even}E) \otimes_L W(k)[[v_1, \dots, v_{n-1}]].$$

THis is non-zero since this ring has a homomorphism to k' for some field k'. To give such a homomorphism is the same as to give a maps $\pi_{even} E \to k'$ and $W(k)[[v_1, \dots, v_{n-1}]] \to k'$ and an isomorphism of corresponding formal groups over k'. This can be accomplished for a suitably large field. □

[[★★★ Is this as spectra or as *ring* spectra?]]

Corollary 17.11. *$K(n) \simeq K(n)'$ for any other choice of $K(n)'$.*

Example 17.12. $n = 1$. $f(x, y) = x + y + xy$ over $\mathbb{F}_p$. Then $W(\mathbb{F}_p) = \mathbb{Z}_p$ the p-adic integers. Any deformation of this is a universal deformation. [[★★★ check this]]. Then $\pi_* E(1) \simeq \mathbb{Z}_p[\beta^{\pm 1}]$, with FGL $f(x, y) = x + y + \beta xy$. We already know a spectrum like this. We can start with complex K-theory K, where $\pi_* K = \mathbb{Z}[\beta^{\pm 1}]$. ◇

Corollary 17.13. *$E(1) \simeq \hat{K}$ (the p-adic completion of K-theory).*

Morava Stabilizer group. the Endomorphisms of f is $\mathbb{Z}_p$, and the automorphisms are $\mathbb{Z}_p^\times$. The action on $\hat{K}$ is via Adams operations. There exists a smaller subgroup $\mu_{p-1} \subset \mathbb{Z}_p^\times$. This acts on $\hat{K}_*$. We can consider a new cohomology theory given by,

$$X \mapsto (\hat{K}_* X)^{\mu_{p-1}}.$$

This actually does produce a cohomology theory which is represented by $\hat{K}^{adams}$. This is called the *Adams summand of* $\hat{K}$. $\pi_* \hat{K} = \mathbb{Z}_p[\beta^{\pm 1}]$, and μ_{p-1} acts by the identity character on β. Thus $\pi_* \hat{K}^{adams} \cong \mathbb{Z}_p[\beta^{\pm(p-1)}]$.

If we did this carefully, we could do it as a structured ring spectrum. Then we can get,

$$\hat{K}^{adams} \xrightarrow{p} \hat{K}^{adams} \to \hat{K}^{adams}/p$$

This quotient can also be made into a structured ring spectrum. It is automatically complex orientable. Since for $f(x, y) = x + y + \beta xy$ we have $[p](t) = \beta^{p-1} t^{p?}$, we get that $K(1) \simeq \hat{K}^{adams}/p$.

Starting next week, we will start the nilpotence theorem.

18 The Nilpotence Theorem

Last time we introduced when a spectrum E was a *field*. It forces a Küneth formula for computing the E homology of a space. Last time we claimed that the Morava K-theories were essentially the only fields.

Let $f : X \to Y$ be a map. When is f null-homotopic? One approach is to look at $f_* : \tilde{E}_*X \to \tilde{E}_*Y$. If f is null-homotopic, then $f_* = 0$. This can't give a positive answer. If f is stably null-homotopic, then $f_* = 0$. Is there a converse for finding when f is stably null-homotopic?

No. Say E is a field. Then,

$$\tilde{E}_*(X)^{\otimes n} = \tilde{E}_*(X^{\wedge n}) \to \tilde{E}_*(Y^{\wedge n}) \simeq \tilde{E}_*(Y)^{\otimes n}.$$

Corollary 18.1. *If E is a field, then $f : X^{\wedge n} \to Y^{\wedge n}$ is null-homotopic (or stably null-homotopic) then $f_* : \tilde{E}_*(X) \to \tilde{E}_*(Y)$ is zero.*

Taking all this into account, we do sort of get a converse.

Proposition 18.2. *Suppose we are given ring spectra $\{E^\alpha\}$. Then TFAE:*

1. *If R is a ring spectrum $x \in \pi_0(R)$ such that $x \mapsto 0$ in $E^alpha_0(R)$, for all α then x is nilpotent.*
2. *Same condition, but $x \in \pi_n R$.*
3. *X is a spectrum, $x \in \pi_0 X$, $x \mapsto 0 \in E^\alpha_0(X)$ for all α, then $x^{\otimes n} = 0 \in \pi_0 X^{\otimes n}$ for sufficiently large n.*
4. *If F is a finite spectrum $f : F \to X$ induces a null-homotopy $F \to E^\alpha \otimes X$ for all α, then $F^{\otimes n} \to X^{\otimes n}$ is zero for sufficiently large n.*

Proof. (2) $\Rightarrow$ (1), and (4) $\Rightarrow$ (3). Now (1) $\Rightarrow$ (3): take $R = \oplus_n X^{\otimes n}$. (3) $\Rightarrow$ (4): Replace X by $DF \otimes X$, where DF is the Spanier-Whitehead dual of X. Finally, (4) $\Rightarrow$ (2): Say $x \in \pi_a R$, and it maps to zero in $E^\alpha_a(R)$. Then $x : S^a \to R$, implies by (4), that

$$x^{\otimes n} : S^{na} \to R^{\otimes n}$$

is zero, so $x^n : S^{na} \to R^{\otimes n} \to R$ is zero. □

The Nilpotence Theorem:

Theorem 18.3 (Devinatz-Hopkins-Smith)**.** *MU detects nilpotents; (Equivalently, for any ring spectrum R, the kernel of $\pi_* R \to MU(R)$ consists of nilpotents).*

Corollary 18.4 (Nishida)**.** *For all $n \geq 0$, $x \in \pi_n \mathbb{S}$, then x is nilpotent.*

Corollary 18.5. *The collection of spectra $\{K(n)\}$ detects nilpotence where $0 \leq n \leq \infty$ and we look at all primes. (Recall $K(\infty) = H\mathbb{F}_p$ and $K(0) = H\mathbb{Q}$.)*

Corollary 18.6. *If E is any non-zero ring spectrum, then $E \otimes K(n) \neq 0$ for some n.*

Proof. $1 \in \pi_* E$. If $E \otimes K(n) \simeq 0$ for all n and p, then 1 must be nilpotent, hence it is zero, hence $E \simeq 0$. □

Corollary 18.7. *Let E be a field. Then E is a $K(n)$-module for some (unique) n.*

Proof. There exists an n, such that $E \otimes K(n) \neq 0$. This contains E as a summand[[★★★ Why??]]. Then E is a $K(n)$-module. Say $m \neq n$, then $E \otimes K(m)$ is a $K(n) \otimes K(m)$-module. Hence $E \otimes K(m)$. □

Proof. (of Corollary 18.5) Say X is a spectrum (p-locally), $x \in \pi_0 X$ has total image in $K(m)_0(X)$ for all m. We want $x^{\otimes n} = 0 \in \pi_0(X^{\otimes n})$ for all sufficiently large n.

Take

$$T = \operatorname{colim} S \xrightarrow{x} X \xrightarrow{id \otimes x} X^{\otimes 2} \to X^{\otimes 3} \to \cdots$$

Claim 18.8. For any ring spectrum E, TFAE

1. For sufficiently large n, $x^{\otimes n} \mapsto 0 \in E_0(X^{\otimes n})$,
2. $E \otimes T \simeq 0$

◇

Proof. $E \otimes T = \operatorname{colim} E \xrightarrow{x^{\otimes n}} E \otimes X^{\otimes n}$ [[★★★ each of these is zero?]] so the colimit is zero.

Now assume that $E \otimes T \simeq 0$. This implies that $S \to E \to E \otimes T$ is zero, and hence it must be zero on some term of the colimit. This is precisely condition (1). □

Recall, the we have $M(k)$ which is the cofiber of $t_k : \Sigma^{2n} MU_{(p)} \to MU_{(p)}$.

Definition 18.9. For $m \geq 0$, let $P(m)$ be the smash product over $MU_{(p)}$ of $\{M(k)\}$ where $k \neq p^a - 1$, and also the $\{M(p^a - 1)\}$ for all $a < m$. ◇

We have $\pi_* P(m) = \mathbb{Z}_{(p)}[v_1, v_2, \dots]/(v_0, v_1, \dots, v_{m-1})$.

Example 18.10. $P(\infty) \simeq H\mathbb{F}_p \simeq K(\infty)$. $P(0) \simeq BP$ Landweber exact, and $\operatorname{Spec} \pi_* BP \to \mathcal{M}_{\mathrm{FG}} \times \operatorname{Spec} \mathbb{Z}_{(p)}$ is faithfully flat. In terms of Bousfield classes, $BP \sim MU_{(p)}$. ◇

To prove that x is nilpotent, it suffices to show that $T \wedge BP \simeq 0$. THis is equivalent to $T \wedge MU_{(p)} \simeq 0$ which is equivalent, by the DHS Nilpotence theorem, to x being nilpotent.

So we just need $P(0) \wedge T \simeq 0$. We have $x \mapsto 0 \in P(\infty)_0 X = \operatorname{colim} P(m)_0(X)$. Hence $x \mapsto 0$ in $P(m)_0(X)$ for sufficiently large m. This shows that $P(m) \otimes T \simeq 0$ for sufficiently large m. Goal: prove that $T \otimes P(a) \simeq 0$ for all $a \leq m$. The proof will be by descending induction.

Assume that $P(a+1) \otimes T \simeq 0$. $\Sigma^{2(p^a - 1)} P(a) \xrightarrow{v_a} P(a) \to P(a+1)$. [[★★★ missed something...]]. corollary: $P(a) \otimes T \simeq P(a)[v_a^{-1}] \otimes T$. We want this to be zero.

$P(a)[v_a^{-1}]$ is a module over $MU_{(p)}[v_a^{-1}]$. It suffices to prove that $T \otimes MU_{(p)}[v_a^{-1}] \simeq 0$, and since $MU_{(p)}[v_a^{-1}] \sim E(a)$, it suffices to show that $T \otimes E(a) \simeq 0$.

We will prove this by *ascending* induction. We will show that $E(b) \otimes T \simeq 0$ for all b. Assume we have $E(b) \otimes T = 0$, and then $K(b+1) \otimes T = 0$, since $x \mapsto 0 \in K(b+1)X$. This implies that $E(b+1) \otimes T \simeq 0$ [[★★★ why?]] □

The philosophy of this course is to look at 'Spec $\mathbb{S}$'. We look at Spec $MU \to$ Spec $\mathbb{S}$, and the maps to Spec $\mathbb{S}$ which can be studied by descent from Spec MU are related to the stack of formal groups. What if Spec $MU \to$ Spec $\mathbb{S}$ is non-surjective? The nilpotence theorem shows that this is in some sense not the case. Warning: there exists a spectrum X (non-connective!) such that $X \not\simeq 0$ and yet $MU_*(X) = 0$. The nilpotence theorem implies X cannot be a ring spectrum.

Recall that $\mathcal{C} \subset Sp$ subcategory of spectra, closed under homotopy colimits, desuspensions, (and gen. by a small subcat), then Bousfield localization gives a functorial fiber sequence,

$$C(X) \to X \to L(X)$$

where $L(X)$ is $\mathcal{C}$-local and $C(X) \in \mathcal{C}$. If $\mathcal{C}$ is generated by finite spectra, then L is smashing. (it is given by smashing with a spectrum, it commutes with homotopy colimits [[★★★ or is it limits??]]).

Fix a prime number p, then a spectrum is p-local if every other prime acts invertibly on the homotopy groups of the spectrum (i.e. they are $\mathbb{Z}_{(p)}$-modules i.e. the spectrum is a $\mathbb{S}_{(p)}$-module spectrum). Now we can replace spectra above by p-local spectra.

Definition 18.11. A subcategory T in finite p-local spectra is said to be *thick* if $0 \in T$, T is closed under fibers and cofibers, and T is closed under retracts. ◇

If $\mathcal{C}$ satisfies the above conditions, then $\mathcal{C} \cap$ finite p-local spectra is a subcategory T which is thick. Conversely let T be thick. Let $\mathcal{C}$ be the smallest collection of p-local spectra containing T, and closed under homotopy colimits. [[★★★ some argument]] Then the localization is smashing.

Lemma 18.12. *Let X be a finite p-local spectrum, then if $K(n)_*X = 0$ then $K(n-1)_*X = 0$.*

Proof. $R = MU_{(p)}$-module obtained by smashing $\{M(k)\}$ with $MU_{(p)}[v_n^{-1}]$. Then $\pi_*R = \mathbb{F}_p[v_{n-1}, v_N^{\pm 1}]$ if $n > 1$ and is $= \mathbb{Z}_{(p)}[v_1^{\pm 1}]$ if $n = 1$. $\pi_0 R = \mathbb{F}_p[t]$, where $t = v_{n-1}^a v_n^{-b}$, where $a(p^{n-1} - 1) - b(p^n - 1) = 0$.

$$R \xrightarrow{t} R \to R/t \simeq R/v_{n-1}^a$$

This last one has a finite filtration by copies of R/v_{n-1}. Then $K(n)_*X = 0$ implies that $(R/t)_*X = 0$. Hence multiplication by t is invertible on $R_*(X)$.

Now suppose that $R_*(X)$ is a finitely generated module over π_*R. Since multiplication by t is invertible, each $R_k(X)$ is a finite torsion module over $\mathbb{F}_p[t]$. This means there is some polynomial $f(t)$ which annihilates $R_k(X)$ for $0 \le k < 2(p^n - 1)$, and hence for all k since $R_*(X)$ is periodic with period $2(p^n - 1)$. Without loss of generality we may assume that $t | f(t)$ (just multiply $f(t)$ by t).

Now consider $R[f(t)^{-1}]$. This is another ring spectrum. Then $R[f(t)]_*(X) = 0$. Thus $f[f(t)^{-1}] \otimes K(n-1)_*X = 0$. We want $K(n-1)_*(X) = 0$. This will be true exactly if $R[f(t)^{-1}] \otimes K(n-1) \neq 0$.

Now $R[f(t)^{-1}] \otimes K(m) \neq 0$ for some m, and has two formal groups (from each factor). Thus it must have height $h = m = n - 1$, because $R[f(t)^{-1}]$ is R localized away from the part where the formal group has height exactly n. □

Definition 18.13. A finite p-local spectrum X has *type* n if $K(n)_*X \neq 0$, and $K(m)_*(X) = 0$ for all $m < n$. ◇

Example 18.14. X has type zero if $H_*(X;\mathbb{Q}) \neq 0$, and $H_*(X;\mathbb{Z})$ is not torsion. ⋄

Note: if X is finite, then $H_*(X;\mathbb{F}_p)$ is concentrated in finitely many degrees. Now there is an Atiyah-Hirzebruch spectral sequence

$$H_a(X;(\mathbb{F}_p[v_n^{\pm 1}])_b) \Rightarrow K(n)_*(X).$$

This degenerates for very large n. Thus $K(n)_*(X) \cong H_*(X;\mathbb{F}_p)[v_n^{\pm 1}]$ for sufficiently large n. Thus $K(n)_*X = 0$ for all n is equivalent to $H_*(X;\mathbb{F}_p) = 0$ is equivalent to $X \simeq 0$. By convention we will say X has type ∞.

Definition 18.15. $\mathcal{C}_{\geq n}$ is finite p-local spectra X of type $\geq n$. It is also those X such that $K(m)_*X = 0$ for $m < n$. ⋄

Observation: $\mathcal{C}_{\geq n}$ is thick.

Theorem 18.16 (Thick Subcategory Theorem)**.** *Every thick subcategory has the form* $\mathcal{C}_{\geq n}$ *for some* $0 \leq n \leq \infty$*.*

we have open loci $\mathcal{M}_{\mathrm{FG}}{}^{\leq n} \subseteq \mathcal{M}_{\mathrm{FG}} \times \operatorname{Spec} \mathbb{Z}_{(p)}$. This later has a structure which is like an infinite flag. The above theorem says the this is a good picture of the category of spectra.

Proof. Case 1: $T = \{0\}$, then $n = \infty$.

Case 2: There exists $x \in T$ where X is non-zero and X has type n for some $n < \infty$. We take n minimal. Then we have $T \subseteq \mathcal{C}_{\geq n}$ by minimality. The hard part is to show that $\mathcal{C}_{\geq n} \subseteq T$, i.e. if Y has type $\geq n$, then $Y \in T$.

Note: If T is thick, then any $X \in T$, then $X \otimes F \in T$ for any finite F. (This follows from the properties of being thick). Let $DX = \mathbb{S}_{(p)}^X$. Then $f : \mathbb{S}_{(p)} \to X \otimes DX$. Note: f induces an injection $K(m)_*\mathbb{S}_{(p)} \to K(m)_*(X \otimes DX)$ for $m \geq n$. This element corresponds to the identity map of $K(m) \otimes X$ and hence it is zero if and only if $X \otimes K(m) \simeq 0$.

Let's consider a map $g : F \to \mathbb{S}_{(p)} \to Y \otimes DY$. Then g is zero on $K(m)_*F$ for $m \geq n$. It is also zero on $K(m)_*F$ for $m < n$. [[★★★ Why do these statements hold?]]

For k sufficiently large, $g^{\otimes k} : F^{\otimes k} \to (DY \otimes Y)^{\otimes k} \to DY \otimes Y$. This map is adjoint to a map $Y \times F^{\otimes k} \to Y$. THis is a composition of ... [[★★★ It is zero?? why?]].

Then $Y/(Y \otimes F^{\otimes n}) \simeq Y \oplus \Sigma(Y \otimes F^{\otimes n})$. We want $Y \in T$. Thus it suffices to show that $Y/(Y \otimes F^{\otimes n})$ is in T. By this has a filtration, with successive quotients $Y \otimes F^{\otimes a}/Y \otimes F^{\otimes a+1} \simeq Y \otimes F^{\otimes a} \otimes (S_p/F) \simeq Y \otimes F^{\otimes a} \otimes (X \otimes DX)$. □

Question: Are these classes of spectra different if $m \neq n$? i.e. does there exist a spectrum of type n for every n?

Answer: Yes. Next time we will sketch a proof, using the v_n-self maps.

So for example when $n = 0$, this is obvious. $X = \mathbb{S}_{(p)}$. When $n = 1$, we can take $X = M(p)$, the mod p Moore space $\mathbb{S} \xrightarrow{p} \mathbb{S} \to M(p)$. We can compute it's $K(1)$-homology with this cofiber sequence. We deduce that $M(p)$ has type one. For $n = 2$, we would like to modify $M(p)$ in some way to make it type 2. For the next step we want a map 'v_1' : $M(p) \to M(p)$. This doesn't quite make sense. What we really want is a map with induces multiplication by v_1 on passing to $K(1)$-homology.

goal: show there exists a spectrum of type n for all n

Idea: induction on n. Assume we have X of type n. Then

$$\Sigma^k \xrightarrow{f} X \to X/f$$

Hope that X/f has type $n+1$. Since type $\geq n$ is thick, X/f has type $\geq n$. To gaurentee X/f is not of type n, f induces $K(n)_*X \simeq K(n)_*X$. To guarentee X/f is not of type $> n+1$, $K(n+1)_*X \to K(n+1)_*X$ is [[★★★ missed this.]]

Definition 18.17. Let f be a (p-local) finite spectrum map $f : \Sigma^k X \to X$ is a v_n*-self map* if

1. $f : K(n)_*X \simeq K(n)_*X$ is an isomorphism,
2. $f : K(m)_*X \to K(m)_*X$ is nilpotent, for $m \neq n$.

◇

Note: If X has a v_n-self map, then the type of X is $\geq n$. Then X/f acyclic for $K(n)_*$ implies it is acyclic for $K(m)_*$ for $M \leq n$, and hence X is acyclic for $K(m)$, for $m < n$.

Theorem 18.18 (Peroidicity Theorem)**.** *Every p-local fintie spectrum of type $\geq n$ admits a v_n-self map.*

Corollary 18.19. *There exist spectra of type n for any n.*

For such X, a map $f : \Sigma^k X \to X$ is the same as $f \in \pi_k R$ where $R = X \otimes DX$.

Definition 18.20. Let R be a p-local finite ring spectrum. A class $x \in \pi_k R$ is a v_n*-element* if $x \mapsto$ nilpotent in $K(m)_*R$, for $m \neq n$ and $x \mapsto$ invertible in $K(n)_*R$. ◇

Note multipliction by x is a v_n-self map if if is a v_n-element.

Lemma 18.21. *If $x \in \pi_k R$ is a v_n-element, then after raising x to a power we can assume that $x \mapsto 0$ in $K(m)_*R$ for $m \neq n$, and $x \mapsto v_n^j \in K(n)_*R$ (image of Hurewicz map).*

Proof. Recall $K(m)_*R \cong H_*(R;\mathbb{F}_p)[v_m^{\pm 1}]$ for a sufficiently large m. Then x maps to something nilpotent in $H_*(R;\mathbb{F}_p)$, and wlog we can assume it maps to zero in $H_*R;\mathbb{F}_p)$. Thus it maps to zero in $K(m)_*(X)$ for all sufficiently large m. Thus there are a finite number of remaining m where the image is nilpotent. Thus we can assume, by raising x to a single large power, that it maps to zero in $K(m)_*R$ for all $m \neq n$.

Now $x \mapsto \alpha \in K(n)_*R$ invertible. Now $K(n)_*R$ is a finite module over $\mathbb{F}_p[v_n, v_n^{-1}]$. So $K(n)_*R/(v_n - 1)$ is finite. Replacing x by a power, we can assume that $x \mapsto 1 \in K(n)_*R/(v_n - 1) \cong \oplus_{0 \leq i < 2(p^n-1)} K(n)_i R$. And there is a map quotient map $K(n)_*R \to K(n)_*R/(v_n - 1)$. [[★★★ finish this...]] □

Lemma 18.22. *Let A be a $\mathbb{Z}_{(p)}$-algebra (not nec. commutative), containing x, y commuting such that $x - y$ is torsion. Then $x^{p^k} = y^{p^k}$ for some sufficiently large positive k.*

Proof. $(x-y)^{p^a} = 0$, and $p^b(x-y) = 0$. Then

$$\begin{aligned} x^{p^{a+b}} &= (y + (x-y))^{p^{a+b}} \\ &= y^{p^{a+b}} + \sum_{0<i\leq p^{a+b}} \binom{p^{a+b}}{i} y^{p^{a+b}-i}(x-y)^i \\ &= y^{p^{a+b}} \end{aligned}$$

□

Lemma 18.23. *let R be a fintie p-local ring spectrum. $x \in \pi_k R$ a v_n-element. Then replacing x by a power, we can assume that $l_x, r_x : \Sigma^k R \to R$ are homotopic, hence $x \in Z(\pi_* R)$.*

Proof. Let $A = R \otimes DR$, then l_x, r_x give elements $\alpha, \beta \in \pi_k A$, which commute. Since $n \geq 1$, this implies that R has type $\geq n \geq 1$, so A has type ≥ 1, so $\pi_* A$ is torsion. Thus $\alpha - \beta$ is torsion in $\pi_* A$.

Claim 18.24. $\alpha - \beta$ is nilpotent in $\pi_* A$ ⋄

Proof. to prove this claim, it suffice to show that $\alpha - \beta \mapsto 0 \in K(m)_* A$ for all m, i.e. left and right multiplication agree on $R(m)_* R$ for all m. [[★★★ This is true.]] □

Thus, we have $\alpha^{p^a} = \beta^{p^a}$ for some sufficiently large a. Replacing x by x^{p^a}, we get $\alpha = \beta$. □

Let R be a finite p-local ring spectrum.

Claim 18.25. If $x \in \pi_k R$ and $y \in \pi_l R$ are v_n-elements, then $x^i = y^j$ for some i, j. ⋄

Proof. WLOG $k = l$, and x, y are central. Then x, y commute, and $x - y$ is torsion (since $\pi_* R$ is torsion), $x - y$ is nipotent (by the Nilpotence theorem) and so $x^{p^a} = y^{p^a}$ for sufficiently large a. □

Let $f : X \to Y$ be a map of finite p-local spectra, admitting v_n-self maps, $\alpha : \Sigma^k X \to X$ and $\beta : \Sigma^{k'} Y \to Y$. Then raising α and β to appropriate powers, we can assume that $k = k'$, and we have,

$$\begin{array}{ccc} \Sigma^k X & \xrightarrow{f} & \Sigma^k Y \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ X & \xrightarrow[f]{} & Y \end{array}$$

$$S^k \to \Sigma^n DX \otimes Y \rightrightarrows DX \otimes Y$$

where these maps are $D\alpha \otimes id_Y$ and $id_{DX} \otimes \beta$. We want these to be equal. We have seen that these are v_n self maps [[★★★ and so taking appropriate powers of α and β make them equal...]]

Corollary 18.26. *Let τ be a class of p-local finite specta which admit v_n-self maps. Then τ is thick. [[★★★ Missed something?]]*

[[★★★ Further argument using cofibers on the last diagram...]]

$\gamma : \Sigma^k(Y/X) \to Y/X$. Claim: γ is a v_n-self map.

It is an isomorphism on $K(n)_*Y/X$, and without loss of generality α and β give zero on $K(m)_*$. A diagram chase shows that γ *squared* gives the zero map. This is good enough.

We also need to prove that τ is closed under retracts. $f : \Sigma^k(X \oplus Y) \to (X \oplus Y)$ is v_n-self maps.

So a reformulation of the periodicity theorem is that τ is the collection of all spectra of type $\geq n$. I.e. the theorem is equivalence to the statement that there exists a type n spectrum X with a v_n-self map. The proof uses some *explicit* construction of some kind. We will not explain this (due to time constraints).

19 ??

Recall $\mathcal{C}_{\geq n+1}$ the category of p-local finite spectra of type $\geq n+1$. The determines a fiber sequence,

$$CX \to X \to LX$$

where CX is a colimit (in fact a direct limit) of spectra in $\mathcal{C}_{\geq n+1}$, and LX is local, i.e. has no maps from spectra of type $\geq n+1$.

Recall... If X is a spectrum of type $\geq n$, then X admits a v_n-self map $f : \Sigma^n X \to X$, and that f is 'asymptotically unique'. The following direct limit doesn't depend on the particular v_n-self map.

$$\operatorname{colim}(X \to \Sigma^{-n} X \to \Sigma^{-2} X \to \cdots) = X[f^{-1}]$$

$X[f^{-1}]$ is independent of f. Moreover we saw that after raising to suitable powers self maps are functorial. This implies that $X[f^{-1}]$ should be functorial in X.

What is this $X[f^{-1}]$? It is the localization LX. Let's prove this. There is a map $X \to X[f^{-1}]$, and the fiber is a colimit of the fibers i.e.

$$\operatorname{colim}(0 \to \sigma^{-n} X/X \to \Sigma^{-2n} X/X \to \cdots)$$

Now each of these elements are cofibers

$$X \overset{f^{a-an}}{\Sigma} X \to \Sigma^{-ak} X/X$$

each of these has type $> n$. So the fiber of $X \to X[f^{-1}]$ is a colimit of spectra of type $> n$.

Claim 19.1. If Y has type $> n$, then any map $g : Y \to X[f^{-1}]$ is null-homotopic. ⋄

Proof. By adjucntion, it is enough to prove that $DY \otimes X[f^{-1}] \simeq 0$. This follows from the next claim:

Claim 19.2. $id_{DY} \otimes f$ is nilpotent. ⋄

Proof. By The NIlpotentce theorem suffices to show that f is trivial on $K(m)$ for $m \leq n$. For $m < n$ this follows by the Künneth formula in $K(m)$ homology and that $K(m)_*(DY) = 0$ for $m < n$. For $n = m$ we have [[★★★ missed this.]] □

□

Definition 19.3. $L(n)$ is the localization with respect to $\mathcal{C}_{\geq n+1}$. ⋄

Suppose X is a finite p-local spectrum. How do we describe $L(n)(X)$?

$L(0)X \simeq \operatorname{colim}(X \xrightarrow{p} X \xrightarrow{p} X \xrightarrow{p} \cdots)$ and the fiber is $\Sigma^{-1}\operatorname{colim}(p^{-n}X/X)$.

What about $L(1)X$? We have,

$$L(1)\operatorname{colim}\Sigma^{-1}(p^{-n}X/X) \to L(1)X \to L(1)L(0)X$$

This last one is just $L(1)L(0)X = L(0)X$ since it is already local. Since $L(1)$ is smashing and commutes with filtered colimits, we have

$$L(1)\operatorname{colim}\Sigma^{-1}(p^{-n}X/X) \simeq \operatorname{colim} L(1)\Sigma^{-1}(p^{-n}X/X) \simeq \operatorname{colim}\Sigma^{-1}(p^{-n}X/X)[v_1^{-1}].$$

We have the fiber $F \to X \to L(1)X$. Let's compare this to the other cofiber sequence we just considered.

[switching to cofibers]

We compare $X \to L(0)X \to \operatorname{colim} p^{-n}X/X$ to $L(1)X \to L(1)L(0)X \to \operatorname{colim}(p^{-n}X/X)/(\text{powers of } v_1)$.

Carrying this out we get that the fiber of $X \to L(n)X$ is

$$\operatorname{colim} \Sigma^n (X/([p^{a_0}, v_1^{a_1}, v_2^{a_3}, \dots, v_n^{a_{n+1}}]))$$

each of the terms of this colimit depend on particular choices of v_i-self maps, but the colimit does not.

Question 19.4. How does this compare to the localizations $L_{E(n)}$?

A finite p-local spectrum X has type $> n$ if and only if $E(n)_*(X) \cong 0$, which implies that

$$K(0)_*X \cong K(1)_*(X) \cong \ldots \cong K(n)_*X \cong 0$$

Now every direct limit of finite type $> n$ spectra is $E(n)$-acyclic and this is equivalent to the fact that $E(n)$-local spectra are $L(n)$-local. This means that $L_{E(n)}$ factors through $L(n)$, i.e.

$$L_{E(n)}X \simeq L_{E(n)}L(n)X.$$

Now what does this say about our picture in terms of the geometry of formal groups?

$$\mathcal{M}_{\mathrm{FG}}{}^{\leq n} \subseteq \mathcal{M}_{\mathrm{FG}} \times \operatorname{Spec} \mathbb{Z}_{(p)} \supseteq \mathcal{M}_{\mathrm{FG}}{}^{>n+1}$$

Conjecture 19.5 (Telescope Conjecture)**.** *$L_{E(n)} \simeq L(n)$.*

Remark 19.6. The telescope conjecture is generally believed to be false. It is true if $n = 1$ [Miller]. $L(n)$ is called telescopic localization. ◇

$$\{X \mid \text{finite}, L_{E(n)}X \simeq 0\} \simeq \mathcal{C}_{\geq n+1}$$

Note: if the telescope conjecture is true, then this implies that $L_{E(n)}$ is smashing. It is also true regardless that $L_{E(n)}$ is smashing. So if the telescope conjecture is true it provides a counter example to the claim that smashing localizations are given by localizing with respect to a class of spectra.

Note: $L(n)X \simeq X \otimes L(n)\mathbb{S}$, and $L(n)\mathbb{S} = \operatorname{colim}(\cdots)$. The fiber sequence we looked at before is just $X \otimes -$ with the sequence $Q \to \mathbb{S} \to L(n)\mathbb{S}$. We jsut need to understand this one sequence.

let $\mathcal{D}$ be the class of p-local spectra X such that $X \simeq L(n)X$. $\mathcal{D}$ is closed under colimits implies that there exists a fiber sequence,

$$D(Y) \to Y \to RY$$

where RY is D-local.

This is analogous, we have $X^{L(n)\mathbb{S}} \to X = X^{\mathbb{S}} \to X^Q = \lim X/(p^{a_0}, v_1^{a_1}, \dots, v_n^{a_n})$. completing with respect to the closed substack is like completing along this ideal...

This second cofiber sequence is equivalent to $DX \to X \to RX$, i.e.

$$X^{L(n)\mathbb{S}} \in \mathcal{D}$$

and X^Q is $\mathcal{D}$-local (recieves no maps from objects of $\mathcal{D}$). What does R look like? This is not a smashing localization. But it has a description which is sort of dual to the description of smashing localization.

Why is $X^{L(n)\mathbb{S}}$ in $\mathcal{D}$? Wel what if Y has type $> n$. Then any map $Y \to X^{L(n)\mathbb{S}}$ is null-homotopic. Since $0 \simeq L(n)Y \simeq Y \otimes L(n)\mathbb{S} \to X$.

Why is X^Q $\mathcal{D}$-local? Well it is $X^Q = \lim X \otimes (\mathbb{S}/(p^{a_0}, v_1^{a_1}, \ldots, v_n^{a_n}))$. It suffices to show that $X \otimes Y$ is cD-local for Y finite of type $> n$. Let $Z \in \mathcal{D}$. We want every map $Z \to X \otimes Y$ to be null. But $L(n)Z \otimes DY \simeq Z \otimes L(n)DY \simeq 0$.

20 ??

Smashing localizations are a subclass of the localization functors L in p-local spectra. These latter are equivalent to subcategories $\mathcal{C}$ of p-local spectra which are closed under shifts and homotopy colimits and are generated by a set. This in turn contains those subcategories generated by finite p-local spectra. These subcategories always give smashing localizations.

Todays question is whether all smashing localizations arise this way. Last time we considered $cC_{\geq n+1}$ which mapped to the telescopic localization functor $L(n)$. Was got a feel for how to compute with these. We will consider a localization L. To get a handle on it let's consider what it does on Morava K-theory.

Lemma 20.1. *L is a localization, then for all n, $L(K(n))$ is either $K(n)$ or 0.*

Proof. $K(n) \to L(K(n))$, so if $L(K(n))$ is non-zero, then it contains a copy of $K(n)$ (possibly shifted). Note if Y is L-local, then $Y \simeq L(Y)$ and all summands of Y are L-local. [This can be checked by looking at the universal property of localization (the sum of two maps which have the universal property have the universal property)] Thus this means that $K(n)$ (a summand of $L(K(n))$) is local. Hence $K(n) \simeq L(K(n))$. □

Lemma 20.2. *Let L be a smashing localization. Then let $E \neq 0$ be a complex oriented ring spectrum whose formal group has height exactly n. Then $LE \simeq 0$ if and only if $LK(n) \simeq 0$.*

Proof. Assume that $LE \simeq 0$. Then we have

$$0 \simeq LE \otimes K(n) \simeq E \otimes LK(n) \simeq \begin{cases} E \otimes K(n) \\ 0 \end{cases}$$

Now since E has height exactly n, $E \otimes K(n) \not\simeq 0$. So we must have $LK(n) \simeq 0$.

Now suppose that $LK(n) \simeq 0$. Then we have $0 \simeq LK(n) \otimes E \simeq K(n) \otimes LE$, and the same argument [[★★★ missed something]] shows that $LE \simeq 0$. □

Lemma 20.3. *Let L be smashing. Say $0 \leq m < n \leq \infty$. If $LK(m) = 0$, then $LK(n) = 0$.*

Proof. For simplicity assume $0 < m < n < \infty$. Now recall $\Sigma^{2k} MU_{(p)} \to MU_{(p)} \to M(k)$. Let R be the smash product (in $MU_{(p)}$-modules) of $MU_{(p)}[v_n^{-1}]$ with $\{M(k)\}$ with $k \neq p^m - 1, p^n - 1$. Note $\pi_* R = \mathbb{F}_p[v_m, v_n^{\pm 1}]$.

Now we have $LK(m) = 0$. Now look at $R[v_m^{-1}]$. This now has height exactly m. Thus $LR[v_m^{-1}] \simeq 0$. But L is smashing, so it commutes with filtered colimits and $R[v_m^{-1}]$ is a filtered colimit. It is a filtered colimit of suspensions of LR. For this to vanish, this means that $v_m^k \mapsto 0 \in \pi_* LR$ for some k.

Now consider $\Sigma^{2(k+1)(p^m-1)} R \overset{v_m^{k+1}}{\to} R \to R/v_m$. Then v_m^k maps to zero in $\pi_* LR/v_m^{k+1}$. But v_m^k does not map to zero in R/v_m^{k+2}. This means that in particular the map $R/v_m^{k+2} \to LR/v_m^{k+1}$ is not an equivalence. Hence R/v_m^{k+1} is not L-local. R/v_m^{k+1} has a finite filtration with a factor which is $K(n) \simeq \Sigma^{2a(p^m-1)} R/v_m$. This can't be not L-local, so $LK(n) \simeq 0$. □

Proposition 20.4. *For L smashing and $n \geq 0$, the following are equivalent:*

1. $LK(n) \simeq 0$
2. $LK(m) \simeq 0$ *for all* $m \geq n$ *(including* $m = \infty$*)*
3. $LX \simeq 0$ *for all finite* p*-local spectra of type* $\geq n$.

Proof. (1) ⇒ (2) from the previous lemma. Let's do (3) ⇒ (1). There exists a spectrum X of type n. Thus X is a non-zero $K(n)$-module. Moreover $LX \simeq 0$, by assumption. This means that $L(X \otimes K(n)) \simeq 0$, hence L kills any summand of $X \otimes K(n)$, hence $LK(n) \simeq 0$.

Now let's prove that (2) ⇒ (3). Let X have type n, and let $R = X \otimes DX$, this is type n since $K(n)_*$ satisfies a Künneth formula and X of type n implies DX is type n. We want $LX \simeq 0$. LX is a module for LR, so it suffices to show that $LR \simeq 0$. We want $L(R) \otimes K(m) \simeq 0$ for all $\infty \geq m \geq 0$. Since L is smashing, we have $R \otimes LK(m) \simeq 0$ for $m \geq n$. Now there is a map $L(R) \otimes K(m) \leftarrow R \otimes LK(m) \simeq 0$ for $m < n$. □

Let L be smashing. Then there are three cases.

(A) $LK(n) \simeq 0$ for all n.

(B) $LK(n) \simeq K(n)$ for all $n < \infty$.

(C) There exists an n such that $LK(n) \simeq K(n)$ and $LK(n+1) \simeq 0$.

In case (A), we have $L\mathbb{S}_{(p)} \simeq 0$, so $LX \simeq X \otimes L\mathbb{S}_{(p)} \simeq 0$ for all X. Now let's look at case (C). We know that all finite spectra of type $> n$ are in the kernel of L.

Lemma 20.5. *let L be a smashing localization such that $LK(n) \simeq K(n)$. Then the kernel of L consists of $E(n)$-acyclic spectra.*

Proof. X satisfies $LX \simeq 0$. Want $E(n) \otimes X \simeq 0$, but this is equivalent to $LK(m) \otimes X \simeq K(m) \otimes LX \simeq 0$ for $m \leq n$. □

This is equivalent to saying that every $E(n)$-local spectra X is L-local.

In case (C), the kernel of L contains all finite spectra of type $\geq n$. Hence it contains the kernel of the telescopic localization $L(n)$. Moreover it is contained in the kernel of $L_{E(n)}$. The telescope conjecture is that $L(n) = L_{E(n)}$ which would imply that $L = L(n) = L_{E(n)}$. Now the *smash product theorem* is that $L_{E(n)}$ is smashing. This means that all such functors L are trapped between these two cases.

In case (B), $LK(n) \simeq K(n)$ for all $n < \infty$. This implies that every $E(n)$-local spectrum is L-local. If we wanted to use this material to study sphere spectrum we have a tower:

$$\mathbb{S}_{(p)} \to \lim(\cdots \to L_{E(2)}\mathbb{S}_{(p)} \to L_{E(1)}\mathbb{S}_{(p)} \to L_{E(0)}\mathbb{S}_{(p)})$$

Theorem 20.6 (Chromatic convergence Theorem)**.** *This map is an equivalence.*

Corollary 20.7. *In case B, $L \cong Id$.*

Proof. Each $L_{E(n)\mathbb{S}_{(p)}}$ is L-local implies that $\mathbb{S}_{(p)}$ is L-local, and since L is smashing $LX \simeq X \otimes L(\mathbb{S}_{(p)}) \simeq X$. □

21 E-Localization

This is in preparation for proving the smash product theorem. Let E be a (structured) A_∞-ring spectrum. To any spectrum X we can construct

$$X \to (E \otimes X \rightrightarrows E^{\otimes 2} \otimes X \Rrightarrow \cdots)$$

denote this cosimplicial spectrum by X^*. THen there is a map $X \to TotX^*$. How close is this to X? If $X \to Y$ is an E-equivalence, then $X^* \to Y^*$ is an equivalence. So at best you aren't going to be able to distinguish X and it's localization. Moreover since each of the terms is E-local, $TotX^*$ is E-local. Thus we have a map $L_E X \to TotX^*$.

Question 21.1. Is this a homotopy equivalence?

Is $E \otimes X \to E \otimes TotX^*$ and equivalence? What about $E \otimes TotX^* \to Tot(E \otimes X^*)$? Neither needs to be an equivalence. However, the composite

$$E \otimes X \to Tot(E \otimes X^*)$$

is an equivalence. There is an augmented simplicial object:

$$X \to (E \otimes X \rightrightarrows E^{\otimes 2} \otimes X \Rrightarrow \cdots)$$

and the object

$$E \otimes X \to (E \otimes E \otimes X \rightrightarrows E \otimes E^{\otimes 2} \otimes X \Rrightarrow \cdots)$$

is a *split* augmented simplicial object. Such an object always exhibits the -1-object as a totalization.

$E \otimes TotX^* \simeq E \otimes \lim Tot^n X^* \to \lim E \otimes Tot^n X^* \simeq \lim Tot^n(E \otimes X^*) \simeq Tot(E \otimes X^*)$.

21.1 Prospectra

Definition 21.2. A prospectrum is a diagram of spectra

$$\cdots \to X(2) \to X(1) \to X(0)$$

$Map(\{X(n)\}, \{Y(m)\}) = \lim_m \operatorname{colim}_n Map(X(n), Y(m))$ ◇

A pro-spectrum $\{X(n)\}$ is pro-constant if it is equivalent to a constant pro-spectrum.

Observation: If $\{X(n)\}$ is pro-constant, then $E \otimes \lim X(n) \to \lim E \otimes X(n)$ is an equivalent. (it is funcotrial in pro-spectra.)

Corollary 21.3. *$L_E X \simeq TotX$ if the tower $\{Tot^n X\}$ is pro-constant.*

How do you recognize a pro-constant tower? Note to any tower $\{X(n)\}$ we can associate a spectral sequence converging to $\pi_* \lim X(n)$. In the special case that the pro-spectrum comes from the totalization of a cosimplicial object coming from E and X you get the Adams-Novikov spectral sequence.

Theorem 21.4 (Bousfield)**.** *Let $\{X(n)\}$ be a tower such that there exists an integer $s \geq 1$ such that for every finite F, the tower $(\cdot \to F \otimes X(1) \to F \otimes X(0))$ has a spectral sequence such that $E_r^{p,q} = 0$ if $p, r \geq s$. Then $\{X(n)\}$ is pro-constant.*

(remark the proof will show that we can weaken this to allow r to depend on p.)

Let $\{E_r^{p,q}, d_r\}$ be the spectral sequence of $\{X(n)\}$.

Notation: If $m \leq n$, let $F(n,m)$ be the fiber of $X(n) \to X(m)$.

Definition 21.5. $E_r^{p,q} = \operatorname{im} \pi_q F(p+r-1, p-1) \to \pi_q F(p, p-r)$. ◇

For example, $r = 1$, $E_1^{p,q} = \pi_q F(p, p-1)$. THis is a sort of associated graded and is the beginning of a spectral sequence. Suppose that $X(p) = 0$ for $p < 0$. Then $F(p, p') = X(p)$ if $p' < 0$. Hence if $r > p$, then $E_r^{p,q} = im(\pi_q F(p+r-1, p-1) \to \pi_q X(p+r-1) \to \pi_q X(p))$ So we have

$$E_r^{p,q} = im(\pi_q X(p+r-1) \cap \ker(\pi_q X(p) \to \pi_q X(p-1))).$$

Moreover the map $d_r : E_r^{p,*} \to E_r^{p+r,*}$. (maybe it lands in $E_r^{p+r-1,*}$...) If $r > s$, $d_r = 0$. If $r > s$, then $d_r = 0$ (maybe $r > s+1$...). This implies that $E_r^{p,q} \cong E_{r'}^{p,q}$ for $r, r' > s$.

This implies that the image of $\ker(\pi_q X(p+r) \to \pi_q X(p-1))$ in $\pi_q X(p)$ is independent of r if $r > p, s$. Now we will consider a similar but stronger condition.

This will also be true for the $\ker(\pi_q X(p+r) \to \pi_q X(p-2)))$ since this later is filered by things covered by the last case. By induction this holds for $\ker(\pi_q X(p+r) \to \pi_q X(p-k)))$. I.e. The image of $\pi_q X(p+r) \to \pi_q(X(p))$ is independent of r, for $r \geq p, s$. Call this common image $A_p \subset \pi_q X(p)$. (fix q for the moment). We get surjective maps,

$$A_p \to A_{p-1}.$$

Note: These are injective when p is large enough, $p \geq s$.

Now we will study a tower of abelian groups. $\cdots \pi_q X(4s) \xrightarrow{\theta} \pi_q X(2s) \to \pi_q X(s)$, and the maps $A_{4s} \to A_{2s} \to A_s$ are isomorphisms. These sit inside the last sequence. $A_{4s} \cap \ker \theta = 0$.

conclusion: If $X = \lim X(n)$ (homotopy limit), then $\pi_q X \simeq A$. Moreover $\pi_q X \to \pi_q X(2^k s)$ is injective, and the image is A_{2^k}.

Replace $\{X(n)\}$ by $\{X(n)/X\}$. The maps on homotopy groups $\pi_* X(2^k s)/X \to \pi_* X(2^{k-1} s)/X$. What we want is that $\{X(n)/X\}$ is a zero.

Let $Y(k) = X(2^k s)/X$. It suffices to show that $\{Y(k)\}$ is trivial (as a pro-object). We know that $F \otimes Y(k) \to Y(k-1) \otimes F$ is zero for all finite F. Thus each $Y(k) \to Y(k-1)$ is a phantom map. The system $\{Y(k)\}$ is trivial because a composition of two phantom maps is zero.

[[★★★ Move this to the phantom map section...]]

Let $f : Y \to Z$ be a phantom map. Let $\oplus F_\alpha \to Y$ (not assumed even). This has a fiber Y'. As before we know that Y' is a retract of a sum of finite spectra. Let's say that $Y \to Z$ is a phantom map. This means that it kills $\oplus F_\alpha \to Y \to Z$, and so it factors $Y \to \Sigma Y' \to Z$. But if $Z \to W$ is phantom, and since Y' is a retract of finite spectra, we have the composite $\Sigma Y' \to Z \to W$ is zero. So phantom maps compose to zero.

Recall, if $\cdots X(2) \to X(1) \to X(0)$ is a tower of spectra, then the Tower is pro-constant if there exists $s \geq 1$ such that for all finite F, the tower $\{X(n) \otimes F\}$ has $E_s^{p,q} = 0$ for $p \geq s$.

Recall what the motivating application is. We have a ring spectrum E and a spectrum X and we were looking that the cosimplicial spectrum $X^* = E^{\otimes *+1} \otimes X$. If the tower $\{Tot^n X^*\}$ is pro-constant, then it has value $L_E X$. In this case, $\{(Tot^n X^*) \otimes Y\}$ is equivalent to $(L_E X) \otimes Y$, but the former is just $\{Tot^n (X \otimes Y)^*\}$ and since this is pro-constant it is $L_E(X \otimes Y)$. Altogether this implies that the canonical map $(L_E X) \otimes Y \to L_E(X \otimes Y)$ is an equivalence.

Let τ be the collection of app p-local finite spectra such that $\{Tot^n X^*\}$ is pro-constant. Note that τ is thick. Hence if τ contains any spectrum of type zero then it contains $\mathbb{S}_{(p)}$. In this case (assuming E is p-local),

$$L_E Y \simeq (L_E \mathbb{S}) \otimes Y$$

i.e. L_E is a smashing localization.

Goal: Prove the smash product theorem: that $L_{E(n)}$ is smashing. It suffices to show that there exists a type zero spectrum X such that there exists $s \geq 1$ such that for all finite F, the E-based adams spectral sequence for $X \otimes F$ has $E_2^{p,q}$ vanishing for $p \geq s$. (we really just need $E_s^{p,q} = 0$).

Recall: the adams spectral sequence for X has $E_2^{p,q}$ given by the cohomologies

$$E_*(X) \to (E \otimes E)_*(X) \to (E \otimes E \otimes E)_* X \to \cdots$$

Here $E = E(n)$, and $\operatorname{Spec} \pi_* E(n) \to \mathcal{M}_{\mathrm{FG}}$ is a flat map. These cohomologies are the pullbacks of the coherent sheaf $\mathcal{F}_X$ to $\operatorname{Spec} \pi_* E(n)$ and it's iterated fiber products. Let $\mathcal{F}_X$ denote the sheaf on $\mathcal{M}_{\mathrm{FG}}$ corresponding to X. We have,

$$E_2^{p,q} = H^p({\mathcal{M}_{\mathrm{FG}}}^{\leq n}; \mathcal{F}_X|_{{\mathcal{M}_{\mathrm{FG}}}^{\leq n}}).$$

Want: A type zero spectrum X and $s \geq 1$ such that for all finite Y,

$$H^p({\mathcal{M}_{\mathrm{FG}}}^{\leq n}; \mathcal{F}_{X \otimes Y}) = 0$$

for $p \geq s$. ($\mathcal{F}_{X \otimes Y}$ denotes the restriction of $\mathcal{F}_{X \otimes Y}$ to ${\mathcal{M}_{\mathrm{FG}}}^{\leq n}$). We need to control these cohomologies.

Let's assume that X is *even*. That is $H_*(X; \mathbb{Z}_{(p)})$ is torsion free and of even degrees. This is equivalent to the statement that X has a finite cell decomposition with cells which are even shifts of $\mathbb{S}_{(p)}$. This implies that the AHSS degenerates and hence $MU \otimes X$ is a free $MU_{(p)}$-module. Hence we have a Künneth formula,

$$MU_*(X \otimes Y) \cong MU_*(X) \otimes_{\pi_* MU} MU_*(Y).$$

This implies that $\mathcal{F}_X \otimes \mathcal{F}_Y \simeq \mathcal{F}_{X \otimes Y}$.

So we will construct a non-zero even spectrum X, such for all quasi-cohenrent sheaves $\mathcal{G}$ on ${\mathcal{M}_{\mathrm{FG}}}^{\leq n}$,

$$H^p({\mathcal{M}_{\mathrm{FG}}}^{\leq n}; \mathcal{F}_X \otimes \mathcal{G}) = 0$$

for $p \geq s$. In particular this is sufficient by looking at $\mathcal{G} = \mathcal{F}_Y$.

For $0 \leq k \leq n$, let ${\mathcal{M}_{\mathrm{FG}}}^{\leq n, \geq k}$ be closed substack of ${\mathcal{M}_{\mathrm{FG}}}^{\leq n}$. Idea: prove that

$$H^p({\mathcal{M}_{\mathrm{FG}}}^{\leq n, \geq k}; \mathcal{F}_X \otimes \mathcal{G}) = 0$$

by descending induction. Note: it is true for $k > n$.

Induction step: ${\mathcal{M}_{\mathrm{FG}}}^{\leq n, \geq k+1} \to {\mathcal{M}_{\mathrm{FG}}}^{\leq n, \geq k}$ is a closed substack cut out by the section $v_k \in \Gamma(\omega^{p^k-1})$. We have

$$0 \to K \to \mathcal{G} \xrightarrow{v_k} \mathcal{G} \otimes \omega^{p^k-1} \to K' \to 0$$

where K, K' are supported on ${\mathcal{M}_{\mathrm{FG}}}^{\leq n, \geq k+1}$.

We know [[★★★ assume?]] that $\mathcal{F}_X$ is a vector bundle. Thus,

$$H^p({\mathcal{M}_{\mathrm{FG}}}^{\leq n, \geq k}; \mathcal{F}_X \otimes K) \cong {\mathcal{M}_{\mathrm{FG}}}^{\leq n, \geq k}; \mathcal{F}_X \otimes K') \cong 0$$

for p sufficiently larger then 0. This implies that multiplication by v_k is an isomorphism

$${\mathcal{M}_{\mathrm{FG}}}^{\leq n, \geq k}; \mathcal{F}_X \otimes \mathcal{G}) \to {\mathcal{M}_{\mathrm{FG}}}^{\leq n, \geq k}; \mathcal{F}_X \otimes \mathcal{G} \otimes \omega^{(p^k-1)}) \to {\mathcal{M}_{\mathrm{FG}}}^{\leq n, \geq k}; \mathcal{F}_X \otimes \mathcal{G} \otimes \omega^{2(p^k-1)})) \to \cdots$$

These are all isomorphism. The direct limit of the cohomologies is then $H^p({\mathcal{M}_{\mathrm{FG}}}^{k}; \mathcal{F}_X \otimes \mathcal{G})$.

So we want a non-zero even p-local spectrum X, and $s \geq 1$, such that $H^p({\mathcal{M}_{\mathrm{FG}}}^{k}; \mathcal{F}_X \otimes \mathcal{G})$ vanishes for $p \geq s$ and all $\mathcal{G}$. For $k > 0$ we have

$$\begin{array}{ccc} \operatorname{Spec} \overline{\mathbb{F}}_p \otimes BG_k & \to & {\mathcal{M}_{\mathrm{FG}}}^{k} \\ \downarrow & \ulcorner & \downarrow \\ \operatorname{Spec} \overline{\mathbb{F}}_p & \longrightarrow & \operatorname{Spec} \mathbb{F}_p \end{array}$$

where G_k is the automorphism group of a FGL of height k over $\overline{\mathbb{F}}_p$. Thus we have, by base change,

$$H^p({\mathcal{M}_{\mathrm{FG}}}^{k}; \mathcal{F}_X \otimes \mathcal{G}) \otimes \overline{\mathbb{F}}_p \cong H^*_{cts}(G_k; \mathcal{F}_X \otimes \mathcal{G})$$

where this is pro-finite group cohomology. For any finite even spectrum X, $\mathcal{F}_X$ gives and $\overline{\mathbb{F}}_p$-vector space (basically it is $W_X \cong \overline{\mathbb{F}}_p \otimes (K(n)_* X/(v_n - 1)))$ with continuous action of $G_k = End(f)^\times$ (this is close the the Morava stabilzer group, but over $\overline{\mathbb{F}}_p$). We want to choose X such that for all representations V of G_k, we have,

$$H^p(G_k; W_X \otimes_{\overline{\mathbb{F}}_p} V) \cong 0.$$

Let f be a FGL of height k over $\overline{\mathbb{F}}_p$. Then $G_k \cong End(f)^\times$. And $End(f)$ is a valuation ring of rank k^2 as a $\mathbb{Z}_p$-module. So G_k has the structure of a p-adic Lie group. Here is another p-adic Lie group $(\mathbb{Z}_{(p)}^{k^2}, +)$. There is a filtration of G_k

Theorem 21.6 (Lazard ?)**.** *For any p-adic Lie group there is any open subgorup U, such that $H^p(U; V) \cong 0$ for $p > k^2$.*

Theorem 21.7 (Serre)**.** *The same is true for the group G_k itself if G_k has no elements of order p. [[★★★ This works for more general p-adic Lie groups.]]*

Elements of order p in G_k are the same as p^{th}-roots of unity in the division algebra $D = End(f)[p^{-1}]$. So we would get $\mathbb{Q}_p(\xi_p) \to D$, which exists if and only if $(p-1)|k$.

Corollary 21.8. *$L_{E(n)}$ is smashing if $n < p$, (and we don't have to choose X carefully).*

More generally we won't have $H^p(G_k; V) \cong 0$ for large p for *any* V, but we don't need that. We just need it to be zero for $W_X \otimes V$ (and large p). If there are elements of order p, then there is only one up to conjugacy. Call it u. We need $u^{\mathbb{Z}/p}$ to act freely on $W_X \otimes V$.

You can do this, but we would need to dig too deep into modular representation theory.

What we need is that [[★★★ Missed this...]]

22 Chromatic Convergence Theorem

let X be a spectrum. We can associate the Chromatic tower,

$$\cdots \to L_{E(2)}X \to L_{E(1)}X \to L_{E(0)}X$$

X maps to the limit.

Theorem 22.1. *If X is a finite p-local spectrum, then the map from X to the limit of this tower is an equivalence.*

If suffices to prove this for $X = \mathbb{S}_{(p)}$. We have $C_nX \to X \to L_{E(n)}X$, and so we just need to prove that the limit of $C_n\mathbb{S}_{(p)}$ is $\simeq 0$.

We will use: each of the maps $MU_*(C_{n+1}\mathbb{S}_{(p)}) \to MU_*(C_n\mathbb{S}_{(p)})$ is zero. We will prove this next time. We wold like to leverage this into some information before we take MU-homology.

Consider the unit map $S \to MU$, let the homotopy fiber be I. Since $\mathbb{S}$ is a ring, we get,

$$I^{\otimes n} \to \mathbb{S}$$

and hence

$$I^{\otimes n} \otimes X \to X$$

Definition 22.2. $x \in \pi_*X$ has Adams-Novikov filtration $\geq n$ if it is in the image $\pi_*(I^{\otimes n} \otimes X)$. ◇

There is a cofiber sequence $I^{\otimes n} \otimes X \to X \to Tot^{n+1?}(X^*)$ (the totalization of the Adams-Nov. resolution X^*).

Lemma 22.3. *Let $f : X \to Y$ be a map of spectra such that f induces the zero map on MU-homology, then f increases Adams-Novikov filtration.*

To prove this we need that $I^{\otimes} \otimes f$ also induces zero on MU-homology

Proof. If $x \in \pi_*X$ has filtration $\geq n$, then ... [[★★★ diagram chase..]] □

$\mathbb{S} \to MU \to \Sigma I$, $MU_*MU = (\pi_*MU)[b_i]$, so $MU \otimes I$ is a free MU-module. Hence we have a Künneth formula. [[★★★ I didn't quite follow this.]]

Corollary 22.4. *Each of the maps $\pi_*C_{n+s}\mathbb{S}_{(p)} \to \pi_*C_n\mathbb{S}_{(p)}$ increases Adams-Novikov filtration by s.*

Lemma 22.5. *For each n and m, the Adams-Novikov filtration on $\pi_m C_n \mathbb{S}_{(p)}$ is finite.*

i.e. for s sufficiently large, any element of filtration $\geq s$ is zero.

Thus $\{\pi_m C_n \mathbb{S}_{(p)}\}$ is trivial as a pro-abelian group. This implies that the limit of these groups is zero. We will actually prove:

Lemma 22.6. *If X is connective, the for each n and m, the Adams-Novikov filtration on $\pi_m C_n X$ is finite.*

Definition 22.7. A map of spectra $f : X \to Y$ is *phantom below n* if for any finite F of dimension $\leq n$, $u : F \to X$, the composition $f \circ u$ is null-homotopic. ⋄

Definition 22.8. A spectrum X is MU-convergent if for all n, there exists an s such that $I^{\otimes s} \otimes X \to X$ is phantom below n. ⋄

This implies that every element of $\pi_n X$ of A-N filtration $\geq s$ is 0.

We wan t: If X is connective, then $C_n X$ is MU-convergent.

Lemma 22.9. *If $f : X \to Y$ is phantom below n, and Z is connective,*

$$X \otimes Z \to Y \otimes Z$$

is phantom below n.

Proof. Now if $\dim F \leq n$, then $F \to X \otimes Z \simeq \operatorname{colim} X \otimes Z_\alpha$, where Z_α is finite connective. Hence we get a lift to some $X \otimes Z_\alpha$. This is the same as a map $F \otimes DZ_\alpha \to X$, which the composite $F \otimes DZ_\alpha \to X \to Y$ the zero map. Hence

$$\begin{array}{ccc} F & \xhookrightarrow{0} & Z_\alpha \otimes Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \otimes Z \end{array}$$

□

Note: that I is connected. $\pi_n I = 0$ for $n \leq 0$. Thus if $X \otimes I^{\otimes s} \to X$ is phantom in $\leq n$, then $X \otimes I^{\otimes 2s} \to X \otimes I^{\otimes s}$ is phantom in $\leq n$.

If $X \to Y \to Z$ is a fiber sequence with X, Z MU-convergent then Y is MU-convergent.

Proof. Fix $n \geq 0$ for sufficiently large s, $I^{\otimes s} \otimes X \to X$, and $I^{\otimes s} \otimes Z \to Z$ are phantom below n. [[★★★ diagram chase]]. □

$C_n X \to X \to L_{E(n)} X$. To show $C_n X$ is MU-convergent, it suffices to show that X and $L_{E(n)} X$ are MU-convergent.

Lemma 22.10. *X connective, then X is MU-convergent.*

We want to prove that $L_{E(n)} X$ is MU-convergent.

Lemma 22.11. *If X is an MU-module, then X is MU-convergent.*

Proof. $I \otimes X \to X \to MU \otimes X$ has a section, so the map $I \otimes X \to X$ is null-homotopic. □

Now we will use the smash product theorem. Let $X^* = E(n)^{*+1} \otimes X$. Recall that $L_{E(n)}X \to \{Tot^n X^*\}$ is an equivalence in pro-spectra. In particular there is a map of pro-spectra back the other way. I.e. there is some $Tot^k X^* \to L_{E(n)}X$ which is a retract.

Thus it suffices to show that $Tot^k X^*$ is MU-convergent. This is a finite homotopy limit of the X^m terms. So it suffices to show $X^m = E(n) \otimes (E(n)^{\otimes m} \otimes X)$ is MU-convergent. But $E(n)$ is complex orientable, hence an MU-module, hence MU-convergent.

Next time, we will show that $MU_*(C_{n+1}\mathbb{S}_{(p)}) \to MU_*(C_n\mathbb{S}_{(p)})$ is zero.

We will prove this by computing exactly what these things are and seeing exactly what the maps are.

Example 22.12. $n = 0$ ($C_{-1}X = X$). Then

$$C_0\mathbb{S}_{(p)} \to \mathbb{S}_{(p)} \to L_{E(0)}\mathbb{S}_{(p)} \simeq \mathbb{S}_{\mathbb{Q}}$$

And so we get,

$$\to (\pi_*MU)_{(p)} \to (\pi_*MU)_{\mathbb{Q}} \to MU_*C_0\mathbb{S}_{(p)} \to \pi_*MU_{(p)} \to (\pi_*MU)_{\mathbb{Q}}$$

but be have $L_{(p)} \to L_{\mathbb{Q}}$ is an inclusion and hence $MU_*C_0\mathbb{S}_{(p)} \to \pi_*MU_{(p)}$ is zero and $MU_*C_0\mathbb{S}_{(p)} \cong L_{\mathbb{Q}}/L_{(p)}$. ◇

$L_{(p)} \cong \mathbb{Z}_{(p)}[t_1, t_2, \ldots]$. and we have $t_{p^k-1} = v_k$.

Definition 22.13. $I(n)$ is the ideal in $L_{(p)}$ generated by $(v_0, v_1, v_2, \ldots, v_n)$. An $L_{(p)}$-module M is $I(n)$-torsion if for all $x \in M$ we have $I(n)^k x = 0$ for suff. large k. ◇

[[★★★ missed geometric picture]]

Lemma 22.14. *Let X be an MU-module. π_*X is an L-module. If π_*X is $I(n)$-torsion, then $E(n)_*X \simeq 0$.*

Proof. $E(n)_*X \cong \pi_*E(n) \otimes_{\pi_*MU} MU_*(X)$. Now $MU_*(X)$ is an $\pi_*MU \otimes MU$-module. There are two natural maps $L \rightrightarrows \pi_*MU \otimes MU$, so a priori two ideals corresponding to $I(n)$. However they in fact form the same ideal [[★★★ Same formal groups]]. Moreover $MU_*(X) \cong \pi_*X[b_1, \ldots]$. Thus $MU_*(X)$ is $I(n)$-torsion. Now the substack corresponding to $E(n)$ misses the one cut out by $I(n)$, thus $E(n)_*X \cong 0$. □

Now suppose that X is $I(n-1)$-torsion. Then the above argument doesn't quite work, but what we see is that

Claim 22.15. $L_{E(n)}X \cong X[v_n^{-1}]$. ◇

Proof. Let $\alpha : X \to X[v_n^{-1}]$. We want $X[v_n^{-1}]$ to be $E(n)$-local, and also that is an $E(n)$-homology equivalence. Now $X[v_n^{-1}]$ is a $MU_{(p)}[v_n^{-1}]$-module, hence $MU_{(p)}[v_n^{-1}]$-local, hence it is $E(n)$-local since $E(n)$ is Bousfield equivalent to $MU_{(p)}[v_n^{-1}]$.

Now $X[v_n^{-1}]$ is the colimit of $\Sigma^{-2k(p^n-1)}X$, so if suffices to show that $v_n^k : X \to \Sigma^{-2k(p^n-1)}X$ is an $E(n)$-homology isomorphism for all n. Let the fiber be F. Then π_*F is $I(n)$-torsion module over L. This is becasue v_n act nilpotently and because it is already $I(n-1)$ torsion. □

Notation: $M(n) = L_{(p)}[v_0^{-1}, v_1^{-1}, \ldots, v_n^{-1}]/\sum L_{(p)}[v_0^{-1}, v_1^{-1}, \ldots, v_{i-1}^{-1}, v_{i+1}^{-1}, \ldots, v_n^{-1}]$. Note we also have the inductive formula: $M(n)M(n-1)[v_n^{-1}]/M(n-1)$.

Proposition 22.16. *$MU_*(C_n\mathbb{S}_{(p)}) \cong M(n)$ as an L-module.*

Proof. Induction on n. The case $n = -1, 0$ we have already done, but let's look at them: $\pi_* MU_{(p)} \cong M(-1)$, and $\pi_* C_0\mathbb{S}_{(p)} \cong L_{\mathbb{Q}}/L_{(p)}$.

We have the fiber sequence $C_{n-1}\mathbb{S}_{(p)} \to \mathbb{S}_{(p)} \to L_{E(n-1)}\mathbb{S}_{(p)}$. Let's apply C_n to this. We get,

$$C_n(C_{n-1}\mathbb{S}_{(p)}) \xrightarrow{\simeq} C_n\mathbb{S}_{(p)} \to 0$$

(since $L_{E(n-1)}\mathbb{S}_{(p)}$ is already $E(n)$-local). Now we smash with MU,...

$$MU \otimes C_n\mathbb{S}_{(p)} \to MU \otimes C_{n-1}\mathbb{S}_{(p)} \to MU \otimes L_{E(n)}C_{n-1})\mathbb{S}_{(p)} \simeq L_{E(n)}(MU \otimes C_{n-1}\mathbb{S}_{(p)}).$$

X is an MU-module, $\pi_* X \simeq MU_*(C_{n-1}\mathbb{S}_{(p)}) \cong M(n-1)$ is $I(n-1)$-torsion. Thus we have

$$MU \otimes C_n\mathbb{S}_{(p)} \to MU \otimes C_{n-1}\mathbb{S}_{(p)} \to MU \otimes L_{E(n)}C_{n-1})\mathbb{S}_{(p)} \simeq L_{E(n)}(MU \otimes C_{n-1}\mathbb{S}_{(p)}).$$

becomes

$$??? \to M(n-1) \to M(n-1)[v_n^{-1}]$$

THis is injective, hence the map from $??? = MU_*(C_n\mathbb{S}_{(p)})$ to $M(n-1) = MU_*(C_{n-1}\mathbb{S}_{(p)})$ is zero. □

With the rest of our time, we will discuss a similar kind of argument. Recall that we have $L_{E(n)}$ (which kills all $E(n)$-acyclic spectra). We also have the telescopic localization $L(n)$, which by construction kills finite $E(n)$-acyclic spectra. Thus for all X we have $\alpha : L(n)X \to L_{E(n)}X$.

Claim 22.17. α is an MU-homology equivalence. ◇

(This means the the difference between these (if any) cannont be detected by thinking about the moduli stack of FGs.)

Proof.

$$MU \otimes X \xrightarrow{\beta} MU \otimes L(n)X \to MU \otimes L_{E(n)}X$$

We want to show that β exhibits $MU \otimes L(n)X$ as an $E(n)$-localization of $MU \otimes X$. It is an $E(n)$ homology equivalence since the fiber is built [[★★★ from finite $E(n)$-acyclics??]].

We want that $MU \otimes L(n)X$ to be $E(n)$-local. Idea: use induction on n. We have a fiber sequence,

$$F \to X \to L(n-1)X$$

and the fiber is a finite spectrum of type $\geq n$. smashing with MU we get

$$MU \otimes F \to MU \otimes X \to MU \otimes L(n-1)X$$

Now $MU \otimes L(n-1)X$ is $E(n-1)$-local, hence $E(n)$-local. Now $F =$ the colimit of F_α, where each F_α is finite of type $\geq n$. $MU \otimes L(n)F \simeq \operatorname{colim} MU \otimes L(n)F_\alpha$, so it suffices to show that $MU \otimes L(n)F_\alpha$ is $E(n)$-local [[★★★ recall that these localizations are smashing.]]

So we can assume that X is finite type $\geq n$. There exists a v_n-self map $f : \Sigma^k X \to X$. $L(n)X \simeq X[f^{-1}]$, and we want MU_*X. This is $I(n-1)$-torsion since X has type $\geq n$. Now $MU_*X[f^{-1}] = \pi_* L(n)(X \otimes MU) \to \pi_* L_{E(n)} MU \otimes X \cong MU_*X[v_n^{-1}]$. Is it an isomorphism? Yes.

Proof. $R = X \otimes DX$, and inside $\pi_* MU \otimes R$ we have v_n and f. After raising f to a power, we can assume that $f = v_n^i$
$in \pi_*(MU \otimes R)$. Why is this? Well raising f to a power we can assume that f maps to zero in $K(m)_*R$ for $m \neq n$, and it maps to v_n^i in $K(n)_*R$. We need to show that this is also true in MU-homology, not just Morava K-theory.

[[★★★ mised something. By the Nilpotentce theorem]] it is enough to shoud that $(f - v_n^i)$ maps to zero in $K(m)_*(MU \otimes R)$. When $m < n$ is is clear as well as when $m > n$ [[★★★ Why??]]. When $m = n$, then we get ...

[[★★★ see notes.]] □

□

Recall: The Periodiity theorem says that if X is a finite p-local spectrum of type $\geq n$ then there exists a v_n-self map $f : \Sigma^k X \to X$, and moreover we can take $k = 2(p^n - 1)p^N$. Also these are unique in an asymptotic sense.

[This is the last week of classes, Wednesday is the last class.]

If you recall on the first day of class – asked why is this called Chromatic homotopy theory. I said I would answer it later. Today is later.

Last time we proved the chromatic convergence theorem, that the map from X to the inverse limit of it's chromatic tower,

$$\cdots \to L_{E(1)}X \to L_{E(0)}X$$

is a homotopy equivalence if X is finite p-local. This implies that you get a spectral sequence with $E_1^{p,*} = \pi_*(\text{fiber} L_{E(p)}X \to L_{E(p-1)})$, and moreover this spectral sequence converges reasonably fast (due to pro-constant tower arguments.)

Definition 22.18. $M_n(X) =$ fiber of $L_{E(p)}X \to L_{E(p-1)}$. A spectrum is *monochromatic* (of height n) if X is $E(n)$-local , $E(n-1)$-acyclic. ◇

Example 22.19. $M_n(X)$ is monochromatic. This is clear. In the converse direction, if X is monochromatic, then $X \simeq M_n(X)$. ◇

Let $\mathcal{M}_n$ be the collection of all monochromatic spectra. Observe that this is closed under homotopy colimits.

Definition 22.20. An object $X \in \mathcal{M}_n$ is *compact* if $hom(X, \operatorname{colim} Y_\alpha) \leftarrow \operatorname{colim} \operatorname{Hom}(X, Y_\alpha)$ (these are filtered colimits). ◇

For ordinary spectra this is like be finite. However note that almost no finite spectra are compact in the above sense because they would need to be $E(n)$-local.

Claim 22.21. Let X be a fintie p-local spectrum of type $\geq n$, then $L_{E(n)}X$ is a compact object. ◇

Proof. $L_{E(n)}X$ is $E(n)$-local, and $E(n-1)_* L_{E(n)}X \cong E(n-1)_*X \simeq 0$. Moreover

$$hom(L_{E(n)}X, \operatorname{colim} Y_\alpha) \cong hom(X, \operatorname{colim} Y_\alpha) \cong \operatorname{colim} hom(X, Y_\alpha) \cong \operatorname{colim} hom(L_{E(n)}X, Y_\alpha).$$

□

Let $\mathcal{M}'_n$ be the collection of spectra which are filtered colimits of spectra of the form $L_{E(n)}X$ for X of type $\geq n$. THen $\mathcal{M}'_n \subset \mathcal{M}_n$, and for $Y \in \mathcal{M}_n$ we have,

$$Y' \to Y \to Y''$$

with $Y' \in \mathcal{M}'_n$. $Hom(L_{E(n)}X, Y'') = 0$ for finite X of type $\geq n$. We want $Y'' = 0$.

Proof. $L_{E(n-1)}Y'' \simeq 0$. We will show that $K(n)_*Y'' \cong 0$, hence $E(n)_*Y \cong 0$, and hence $Y'' \simeq 0$. Choose a fintie spectrum of type n. Then $hom(L_{E(n)}DX, Y'') \cong 0 \cong hom(DX, Y'') \cong X \otimes Y''$. Thus we have

$$0 \cong K(n)_*(X \otimes Y'') \cong K(n)_*(X) \otimes_{\pi_* K(n)} K(n)_*(Y)$$

hence $K(n)_*Y \cong 0$ since $K(n)_*X \ncong 0$ by construction. □

conclusion. For all $x \in \mathcal{M}_n$, we can write X as the filtered colimit of X_α, with each $X_\alpha = L_{E(n)}Y_\alpha$, where Y_α is fintie of type $\geq n$. If X is compact, then $X \to \operatorname{colim} X_\alpha$ factors through some X_α, and so X is a retract of X_α.

Corollary 22.22. *$\mathcal{M}_n$ is compactly generated.*

Corollary 22.23. *Compact objects are 'periodic'*

Proof. Let $X \in \mathcal{M}_n$ be compact, X is a summand of $L_{E(n)}Y$, there exists a v_n-self map $f : \Sigma^k Y \to Y$. WlOG $f = v_n^{p^N}$ on $K(n)_*(Y)$, f is an isom on $K(m)_*Y$ for $m < n$ (trivially). This implies that f is an isomorphism on $E(n)_*Y$. Thus,

$$\Sigma^k X \hookrightarrow \Sigma^k L_{E(n)}Y \overset{\simeq}{\to} L_{E(n)}Y \to X$$

also given by $v_n^{p^N}$ on $K(n)_*X$, isomorpic on $K(m)_*X$ for $m < n$. Hence $\Sigma^k X \simeq X$. □

So if $X \in \mathcal{M}_n$, then $X \simeq \operatorname{colim} X_\alpha$ with each X_α periodic with period $2(p^n - 1)p^{N_\alpha}$. This does not imply that X is periodic. But it does mean that the elements of π_*X form 'periodic families'. If $x \in \pi_*X$, then x comes from $\overline{x} \in \pi_k X_\alpha$, and $\overline{x}$ gives classes in $\pi_{k+2(p^n-1)p^{N_\alpha}m} X\alpha \to \pi_{k+2(p^n-1)p^{N_\alpha}m} X$. We will see an example next lecture.

Other things about $\mathcal{M}_n$: If is equivalent to the $K(n)$-local category. With functors

$$L_{K(n)} : \mathcal{M}_n \rightleftarrows Loc_{K(n)} : M_n.$$

Proof. First, say $X \in \mathcal{M}_n$. Then

$$M_n L_{K(n)}X \to L_{E(n)}L_{K(n)}X \to L_{E(n-1)}L_{K(n)}X.$$

The middle term is $L_{E(n)}L_{K(n)}X \simeq L_{K(n)}X$, and since the $L_{E(n-1)}X \simeq 0$, the composition $X \to L_{K(n)}X \to L_{E(n-1)}L_{K(n)}X$ is zero. This means that we have a factorization $X \overset{\alpha}{\to} M_n L_{K(n)}X \to L_{K(n)}X$.

We want to prove that α is an equivalence. X and $M_n L_{K(n)}X$ are $E(n)$-local. SO it suffices to show that $E(n)_*X \cong (E(n))_* M_n L_{K(n)}X$ via alpha. We have that $E(n-1)_*X \cong (E(n-1))_* M_n L_{K(n)}X \cong 0$. So we need $K(n)_*X \cong (K(n))_* M_n L_{K(n)}X$, which is equivalent to $K(n) \otimes L_{E(n-1)}L_{K(n)}X \simeq 0$.

Proof. Note that $E(n-1)$-local spectra are stable under smash product. So $K(n) \otimes L_{E(n-1)Y}$ is $E(n-1)$-local, hence to prove it is zero is suffices to show that $E(n-1) \otimes K(n) \otimes L_{E(n-1)}Y \simeq 0$, but the first part of this smash product is already zero. □

Now for the other direction. We have Y is $K(n)$-local. We want $L_{K(n)}M_n Y \simeq Y$. We have a fiber sequence,

$$M_n Y \to L_{E(n)}Y \to L_{E(n-1)}Y$$

The middle term is just Y. Hence we want to show that $L_{K(n)}L_{E(n-1)}Y \simeq 0$. But this is what we just did. □

Theorem 22.24. *$\mathcal{M}_n$ is equivalent to the category of $K(n)$-local spectra ($\mathcal{K}_n$). Hence $\mathcal{K}_n$ is compactly generated and the compact objects are sumands of $L_{K(n)}X$, for X finite of type $\geq n$.*

Corollary 22.25. *Compact objects of $\mathcal{K}_n$ are periodic.*

Warning 22.26. In general $L_{K(n)}X \in \mathcal{K}_n$ is not compact if X is finite. For example $L_{K(n)}\mathbb{S}$ is not compact. (it's homotopy is not periodic, but it is organized into 'local' periodicity, as expected). ⌟

why 'chromatic homotopy'? The idea is that the chromatic tower

$$\cdots \to L_{E(2)} \to L_{E(1)} \to L_{E(0)}$$

is a 'prism' which outputs M_nX, which is periodic of different periods.

Get the chromatic filtration of π_*X, $F^n\pi_*X = \ker \pi_*X \to \pi_*L_{E(n)}X$. The quotients $F^{n-1}\pi_*X/F^n\pi_*X$ is the E_∞-term of the chromatic spectral sequence. It is not clear that the periodicity survives.

next time we will analyze this for $n = 1$.

23

$\mathbb{S}_{(p)}$ is the limit of the tower $\cdots \to L_{E(1)}\mathbb{S} \to L_{E(0)}\mathbb{S}$. $(\pi_*\mathbb{S})_{(p)}$ is the limits of the homotopy groups of $L_{E(k)}\mathbb{S}$. Look at $n = 1$, $p > 2$.

Goal: inderstand $L_{K(1)}\mathbb{S}$.

Lemma 23.1. *$E(n)$ is $K(n)$-local.*

Notice that $\pi_0E(n) = W(k)[[v_1, \ldots, v_{n-1}]]$, where k is a perfect field of char p.

$F \to \mathbb{S}_{(p)} \to L(n-1)\mathbb{S}_{(p)}$, and the fiber is the colimit of F_α where F_α has type $\geq n$. So $\mathbb{S}_{(p)}$ maps to the 'limit' of DF_α. and we get $\pi_*MU_{(p)} \to MU_*\{DF_\alpha\}$. We can arrange that the DF_α are roughly $\mathbb{S}_{(p)}/(v_0^{N_0}, v_1^{N_1}, \ldots, v_n^{N_n})$.

So what we see is a map $\pi_*MU_{(p)}$ to the inverse system $\{L/(v_0^{N_0}, v_1^{N_1}, \ldots, v_n^{N_n})\}$.

Now we want to do this with $E(n)$ instead of MU. Let $R = W(k)[[u_1, \ldots, u_{n-1}]]$. Then we get a map

$$\pi_*E(n) \cong R \to \{R/(v_0^{N_0}, v_1^{N_1}, \ldots, v_n^{N_n})\}$$

where these v_i are lifts. However these v_i and in the maximal ideal of R. The inverse limit of this system is then R (since it is local) and there is no $\lim^1$-term. Thus we have

$$E(n) \to \lim E(n) \otimes DF_\alpha$$

is an equivalence. So we just need to show that $E(n) \otimes DF_\alpha$ is $K(n)$-local. We know that F_α has type $\geq n$. Let Y be $K(n)$ acyclic. Then $Y \to E(n) \otimes DF_\alpha$ is the same as $Y \otimes F_\alpha \to E(n)$. So it suffices to show that $Y \otimes F_\alpha$ is $E(n)$-local. But $Y \otimes F_\alpha$ is $E(n-1)$-local and $K(n)$-local, so we are done.

Example 23.2. $n = 1$. $E(1)$ is p-adic K-theory. $\hat{K} = \lim K/p^mK$. ◇

notation: Take $E(n)$ to correspond to the FGL f over $\mathbb{F}_{p^n}$ with all endomorphisms defined over $\mathbb{F}_{p^n}$. This means the spectrum $E(n)$ will have lots of symmetry. Let G be the Morava stabilizer group associated to f. We have

$$End^\times(f) \to G \to Gal(\mathbb{F}_{p^n}/\mathbb{F}_p) \cong \mathbb{Z}/n$$

G acts on $E(n)$ in the homotopy category. In fact you can dramatically improve this. G acts continuously on $E(n)$. It is precisely the automorphism group of the E_∞-ring spectrum $E(n)$. Moreover G is a profinite group and the action is continuous.

Fact: $L_{K(n)}\mathbb{S} \simeq E(n)^{hG}$.

The lowbrow approach if $n = 1$ and $p > 2$: In this case $Gal(\mathbb{F}_{p^n}/\mathbb{F}_p) \cong \mathbb{Z}/n = 1$, and $End^{\times}(f) = \mathbb{Z}_p^{\times}$. This group is a direct product of μ_{p-1} and $(1+p\mathbb{Z}_p)^{\times}$. This later is $(\mathbb{Z}_p, +)$. So $\mathbb{Z}_p$ is 'topologically cyclic'. It has a 'generator' g. Let $\psi(g)$ be the automorphism of $E(1)$.

In the lowbrow approach we look at the fiber of $E(1) \stackrel{1-\psi(g)}{\to} E(1)$. This will be our *definition* of $E(1)^{hG}$. Recall that the ψ^g correspond to the Adams operations when we identify $E = \hat{K}$. We have F, the fiber of $(1-\psi^g) : \hat{K} \to \hat{K}$. The map $\mathbb{S} \to \hat{K}$ factors through F. $\alpha : \mathbb{S} \to F$

Claim 23.3. α is an isomorphism on $K(1)$-homology. In particular α induces $L_{K(1)}\mathbb{S} \to F$ is an equivalence. $\diamond$

$\hat{K} = E(1)$ is Landweber exact. So, if we want to compute $\hat{K}_0(\hat{K}/p)$, it is:

$$\pi_0\hat{K} \otimes_L MP_0MP \otimes_L \pi_0\hat{K}/p = R.$$

Then $\pi_0\hat{K} \cong \mathbb{Z}/p$, and $\pi_0\hat{K}/p \cong \mathbb{F}_p$. Then R is the algebra over $\mathbb{F}_p \otimes \mathbb{Z}_p = \mathbb{F}_p$ which classifies automorphisms of the multiplicative formal group. This is the space of continuous maps $\mathbb{Z}_p^{\times} \to \mathbb{F}_p$. Recall that we have a fiber sequence

$$F \to \hat{K} \stackrel{1-\psi^g}{\to} \hat{K}$$

and so ,

$$(\hat{K}/p)_0F \to R \to R$$

where the map $R \to R$ is the identity minus translation by g.

Exercise 23.1. $1 - \psi^g$ is a surjective map from R to R, and the kernel is $\mathbb{F}_p$, the constant functions on $\mathbb{Z}_p^{\times}$.

Thus we get $(\hat{K}/p)_nF \cong \mathbb{F}_p$ if n is even, zero else. This is the same as $(\hat{K}/p)_n\mathbb{S}$. Moreover there is a map $\mathbb{S} \to F$ inducing an isomorphism on these homologies. This means we understand $L_{K(1)}\mathbb{S}$. We have

$$L_{K(1)}\mathbb{S} \to \hat{K} \to \hat{K}$$

and we get a corresponding long exact sequence on homotopy. Recall that $\pi_*\hat{K} \cong \mathbb{Z}_p[\beta^{\pm 1}]$. $\psi^g(\beta) = g\beta$, and it is a ring homomorphism. So this means that $\pi_0 L_{K(1)}\mathbb{S} = \pi_{-1}L_{K(1)}\mathbb{S} \cong \mathbb{Z}_p$. And $\pi_n L_{K(1)}\mathbb{S}$ is zero if $n \neq 0$ is even. Moreover, $\pi_n L_{K(1)}\mathbb{S} \cong \mathbb{Z}_p/(g^k - 1)$ if $n = 2k - 1$.

Now if $p - 1$ does not divide k, then $g^k \not\equiv 1 \bmod p$, and so we get $\mathbb{Z}_p/(g^k - 1) \cong 0$. When $k = (p-1)k'$, then $g^k - 1 \equiv (g^{p-1})^{k'} - 1$. Similar arguments show,$\pi_n L_{K(1)}\mathbb{S} \cong \mathbb{Z}/p^m$ if $n = 2(p-1)p^{m+1}k - 1$, where p does not divide k.

This demonstrates the kind of periodic phenomenon mentioned last time.

$$\begin{array}{ccc} L_{E(1)}\mathbb{S} & \longrightarrow & L_{E(0)}\mathbb{S} \\ \downarrow & & \downarrow \\ L_{K(1)}\mathbb{S} & \to & L_{E(0)}L_{K(1)}\mathbb{S} \end{array}$$

gives us $\pi_n L_{E(1)}\mathbb{S} \cong \pi_n L_{K(1)}\mathbb{S}$ for $n \neq 0, 1, 2$. So we have,

$$\pi_*\mathbb{S}_{(p)} \to \pi_* L_{E(1)}\mathbb{S}$$

claim: This is surjective in positive degrees.

How to check this: Recall the J-homomorphism $j : \pi_* O \to \pi_*\mathbb{S}$. We have $S^k \to O(n)$ which acts on S^n. This gives: $S^{k+n} = S^k \wedge S^n \to S^n$. (stabilize)

Claim 23.4. $j : \pi_* O \to \pi_*\mathbb{S}_{(p)}$ has the same image $Im(j)$, i.e. $Im(j) \cong \pi_* L_{E(1)}\mathbb{S}$. $\diamond$

This story about the image of the j is a philosophical motivator for much of the story of this class.

A The Atiyah-Hirzebruch Spectral Sequence

As far as spectral sequences go, the Atiyah-Hirzebruch spectral sequence is fairly easy to construct. It also has several generalizations, and so it is useful to review how it arises. Also this gives an opportunity to review what a generalized cohomology theory is supposed to be.

Definition A.1. A *reduced cohomology theory* $\tilde{E}$ consists of a sequence of a sequence of functors $E^q : \mathsf{Top}_*^{\mathrm{op}} \to \mathsf{Ab}$ from pointed topological spaces to abelian groups that satisfy the following axioms

1. (Exactness) A $i : A \to X$ cofibration gives an exact sequence
$$\tilde{E}^q(X/A) \to \tilde{E}^q(X) \to \tilde{E}^q(A).$$
2. (Suspension) There are natural isomorphisms
$$\Sigma : \tilde{E}^q(X) \cong \tilde{E}^{q+1}(\Sigma X).$$
3. (Weak Equivalence) If $f : X \to Y$ is a weak equivalence, then $f^* : \tilde{E}^q(Y) \to \tilde{E}^q(X)$ is an isomorphism for all q.
4. (Additivity) If $X = \vee X_\alpha$ is a wedge of a set of (non-degenerately based) spaces, then the inclusions $X_\alpha \to X$ induce an isomorphism
$$\tilde{E}^q(X) \to \prod_\alpha \tilde{E}^q(X_\alpha)$$
for all q.

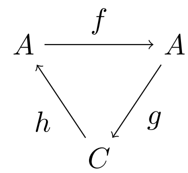
◇

Remark A.2. The exactness axiom applied to the split cofibration sequence $X \to X \vee Y \to Y$ yields a natural isomorphism $\tilde{E}^q(X \vee Y) \cong \tilde{E}^q(X) \oplus \tilde{E}^q(Y)$. Hence the additivity axiom is automatically satisfied for finite wedge products of (nice) spaces. ◇

Unreduced cohomology may be defined as $E^q(X) := \tilde{E}^q(X_+)$, where $X_+ = X \sqcup pt$, is X union a disjoint base point. All the usual axioms of cohomology can be deduced from this. In particular we get a relative cohomology group $E^q(X, A) = \tilde{E}^q(X/A)$ for every pair (X, A) and these fit into the long exact sequence of a pair. Mayer-Vietoris is also satisfied. If $X = \operatorname{hocolim} X_i$ is the homotopy colimit of a sequence of inclusions of spaces, then $E^q(X) = \lim E^q(X_i)$. This is particularly relevant for when a space is given to us with a (nice) filtration $X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X$. In this case the E-cohomology of X may be computed as the limit of the E-cohomology of the individual spaces of the filtration.

One of the most general methods of constructing spectral sequences uses exact couples in an abelian category and is due to Massey. Usually the abelian category in question is the category of doubly graded modules for some ring. However the method of exact couples allows one to construct spectral sequences in more exotic abelian categories, such as the abelian category of doubly graded abelian group valued Mackey functors, which is used in the context of equivariant cohomology.

Definition A.3. An *exact couple* (A, C, f, g, h) consists of a diagram of maps

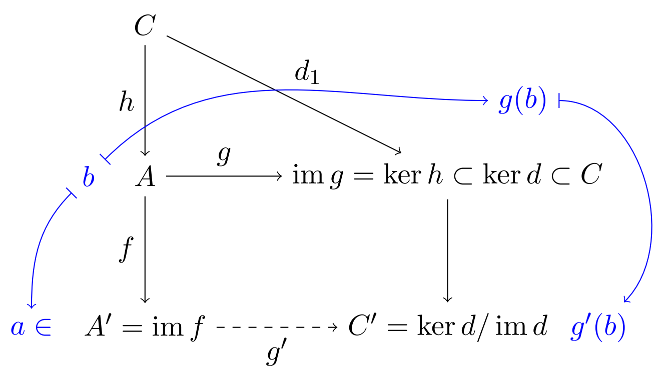


such that each of the sequences is exact

$$A \xrightarrow{f} A \xrightarrow{g} C,$$
$$A \xrightarrow{g} C \xrightarrow{h} A,$$
$$C \xrightarrow{h} A \xrightarrow{f} A.$$

◇

Given an exact couple there is a derived exact couple (A', C', f', g', h') which is defined as follows. First we define a differential $d_1 = g \circ h : C \to C$. Then $A' = \operatorname{im} f$, and C' is the cohomology $C' = \ker d_1 / \operatorname{im} d_1$. The map f' is the restriction of f to $A' = \operatorname{im} f$. The map h' is induced from h (it is easy to check h induces such a map). The map $g' : A' \to C'$ is also induced from g. In an abelian category with elements it is given by a diagram chase as follows.



Given $a \in A' = \operatorname{im} f$, choose $b \in A$, such that $a = f(b)$. Then map it to $g(b) \in C$. It is automatically in $\ker d_1 \subset \ker h$, so descends to an element in $C' = \ker d_1 / \operatorname{im} d_1$. The ambiguity in the choice of b is given by an element $b' \in \ker f = \operatorname{im} h$. Thus there is an element $c \in C$ such that $h(c) = b'$. But then $g(b') = d(c)$, so we get a well defined map $g' : A' \to C'$. The spectral sequence of an exact couple is then obtained by iterating the process of constructing the derived exact couple.

For the Atiyah-Hirzebruch spectral sequence, which incidentally is probably due to Whitehead, we consider a cohomology theory E, and a (nicely) filtered space X. The most important case will be when X is a CW-complex and we use the skeletal filtration. Using the relative cohomology groups and the various long exact sequences obtained from the filtration, we get the following exact couple,

$$\begin{array}{ccc} A = \sum_p E^*(X_p) & \xrightarrow{\quad i^* \quad} & \sum_p E^*(X_{p-1}) = A \\ & {\scriptstyle j^*} \nwarrow \quad \swarrow {\scriptstyle \partial} & \\ & C = \sum_p E^*(X_p, X_{p-1}) & \end{array}.$$

and hence a spectral sequence $E_r^{p,q}$,

$$E_1^{p,q} = E^{p+q}(X_p, X_p - 1) \Rightarrow E^{p+q}(X).$$

In the special case that the filtration $\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X$ is the skeletal filtration of a CW-complex, this simplifies. We have $E_1^{p,q} \cong C^p_{\mathrm{CW}}(X\ \pi_q E)$, the CW-cochains of X with values in the ring $\pi_* E$. The differential can also be checked to agree with the usual cochain differential. This proves the cohomological half of the following theorem.

Theorem A.4. *Suppose that E is a spectrum and X a space with the homotopy type of a CW-complex, then there are half-plane spectral sequences with*

$$E_2^{p,q} \cong H^p(X; E_q(pt))$$
$$E^2_{p,q} \cong H_p(X; E_q(pt))$$

converging conditionally to $E^*(X)$ and strongly to $E_*(X)$, respectively.

Example A.5. $\mathbb{CP}^\infty$ is a CW-complex and it skeletal filtration is given by

$$\mathbb{CP}^\infty_{(-1)} = \emptyset \subset pt \subset \mathbb{CP}^1 \subset \mathbb{CP}^2 \subset \cdots \subset \mathbb{CP}^\infty.$$

This is obviously compatible with the skeletal filtrations of the *finite* CW-complexes $\mathbb{CP}^n$. In each case the Atiyah-Hirzebruch spectral sequence for complex K-theory is concentrated in even degrees. Hence it supports no differential and it collapses at the E_2-term. There is no convergence issue for the finite $\mathbb{CP}^n$, and since everything is free, the extension problem is also trivial. Thus we have,

$$K^*(\mathbb{CP}^n) \cong \mathbb{Z}[\beta, \beta^{-1}][x]/(x^{n+1}).$$

Here $|x| = 2$ and β is the Bott element of complex K-theory. We have $|\beta| = -2$. The K-theory of $\mathbb{CP}^\infty$ can be seen by taking the limit of these groups, which yields,

$$K^*(\mathbb{CP}^\infty) \cong \mathbb{Z}[\beta, \beta^{-1}][[x]].$$

$\diamond$

Let's take a closer look at the passage from the exact couple to the usual description of a spectral sequence as a bunch of pages of doubly graded abelian groups. We mentioned that a standard spectral sequence is obtained by looking at the different (doubly graded) objects $C^{(r)}$ that appear after each iteration of the derived exact couple construction. The new r-page differential can be obtained from the new exact couple. Let us see that it has the correct degree.

Suppose that our abelian category has a notion of *degree*, by which we mean the morphisms form a G-graded abelian group for some abelian group G. In the case of relevance we have $G = \mathbb{Z} \oplus \mathbb{Z}$. Let us further assume that the morphisms of our exact couple are homogeneous, with degrees $|f|$, $|g|$ and $|h|$. Then the differential has degree $|d| = |g| + |h|$. By inspecting the construction of the derived exact couple we see that the new morphisms are again homogeneous and have degrees

$$\begin{aligned} |f'| &= |f|, \\ |g'| &= |g| - |f|, \\ |h'| &= |h|. \end{aligned}$$

Hence the new differential has degree $|d'| = |g| - |f| + |h| = |d| - |f|$.

In the example of the AHSS, the bidegrees (p, q) of the first page of the spectral sequence are given by

$$\begin{aligned} |f| &= (-1, +1), \\ |g| &= (1, 0), \\ |h| &= (0, 0). \end{aligned}$$

Hence the first differential has bidegree $|d_1| = (1, 0)$, and moreover the r^{th}-page has a differential of bidegree $|d_r| = (r + 1, -r)$, as expected.

B Exercises and Solutions

B.1 Exercises (Month One)

Exercise B.1. p^2 does not divide $\binom{p^k}{p^{k-1}}$.

Solution:

Exercise B.2. Moreover if E is compex orientable, then so is $\tau_{\geq 0} E$.

The map $\Omega^{\infty - 2} \tau_{\geq 0} E \to \Omega^{\infty - 2} E$ is an isomorphism on homotopy groups 2 and above.

B.2 Self Directed Exercises

Exercise B.3. Show that the relations defining the Lazard ring L (with generators $a_{i,j}$) are compatible with the grading $|a_{i,j}| = 2(i + j - 1)$.

Solution:

Exercise B.4. Is $K^*(pt)$ a flat L-module?

Solution: No. The map $L \to K^*(pt)$ is given by sending $a_{1,1}$ to a unit in $K^*(pt)$ (typically 1 or β), and by sending all the other generators $a_{i,j}$ to zero. We know by Lazard's theorem that L is abstractly a polynomial algebra on infinitely many generators. This means that it has no torsion, i.e. for any element $x \in L$, the map

$$\cdot x : L \to L$$

is injective. Let x correspond to any element in the kernel of $L \to K^*(pt)$, for example any of the elements $a_{i,j}$ for $(i,j) \neq (1,1)$. Then after tensoring with $K^*(pt)$ we get,

$$K^*(pt) \xrightarrow{0} K^*(pt)$$

which is *not* injective. Hence $K^*(pt)$ is not flat as an L-module.

References

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