Lectures, Seminars, and Courses

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Chapter 1

Equivariant Homotopy Theory and the Kervaire Invariant One Problem

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Abstract: This course will cover the basics of equivariant stable homotopy theory and go through the solution of the Kervaire invariant problem.

1.1 Day One DATE: Aug 31, 2011

The goal is to explain the following theorem

Theorem 1.1.1 (Hill-H.-Ravenel)**.** *If M is a stably framed smooth manifold of Kervaire invariant one, then* $\dim M$ *is one of* $2, 6, 14, 30, 62$ *or* 126*.*

This course will not be about smooth manifold. Homotopy theory provided the tools to solve this.

Started with Pontryagin in the 1930s, with the study of maps $f : S^{n+k} \to S^n$ in terms of $f^{-1}(x)$ for $x \in S^n$. If x is a regular value then this will be a smooth manifold, but it comes with some extra structure. We can choose a basis at the tangent space of x, and this induces a framing of the normal bundle of $M^k = f^{-1}(x) = M_0$. Such an M is what is known as a *stably framed manifold.*

[[★★★ pic of manifold M]]

A different choice of $y \in S^n$ gives a different manifold M_1. In between there is a manifold N with $\partial N = M_0 \sqcup M_1$ as stably framed manifolds. If you push this a little farther, as Pontryagin did, you get a 1-1 correspondence between maps $S^{k+n} \to S^n$ up to homotopy (i.e. $\pi_{n+k}S^n$) and stably framed k-manifolds (+ some conditions) up to cobordism. These conditions are about the n and tell you your bordism is embedded in a sphere S^{n+k}.

Suspension gives a map $\pi_{n+k}S^n \to \pi_{n+k+1}S^{n+1}$, and the limit is $\pi_k^{st}S^0$. We get a bijection between this and stably framed manifolds up to cobordism.

$k = 0$: We get $\pi_0^{st} S^0 = \mathbb{Z}$ counts number of points with signs. It is the degree of the map.

$k = 1$, [[★★★ pic with two framings on the circle]]. $\pi_1^{st} S^0 = \mathbb{Z}/2$. It is not completely obvious how to get this invariant from a random framed 1-manifold.

For $k = 2$, Pontryagin made a very interesting argument. Let's look at oriented surfaces. There is S^2, T^2, and higher genus surfaces. The 2-sphere bounds a disc and the disc has a framing, so there is a framing of the 2-sphere which represents zero. Suppose we have another framing. The there difference is a map $S^2 \to O(n)$, up to homotopy. But this is zero, as was already known in Pontryagin's day (it is a result due to Cartan). You can use the homotopy to construct the cobordism between any two stable framings of S^2.

The next idea introduced by Pontryagin was rediscovered by Milnor in the 50's and is not called (stably framed) surgery. You take your framed surface, you pick an embedded circle, cut open and glue in two discs. It takes something of genus g to genus $g-1$. It lowers the genus by one. We have to make sure this maneuver can allow an extension of the framing. This will be true if the framed circle represents the zero class in $\pi_1^{st} S^0 = \mathbb{Z}/2$. So this gives a map:

$$\varphi : H_1(\Sigma; \mathbb{Z}/2) \to \mathbb{Z}/2$$

and you can do surgery if and only if there is a non-zero element $X \in H_1(\Sigma; \mathbb{Z}/2)$ such that $\phi(X) = 0$.

Pontryagin then made the following argument: If genus of $\Sigma > 0$, then $\dim H_1(\Sigma) > 0$ and is even, hence greater then one, hence the kernel of φ is non-zero and so such an X always exists.

The error is that φ is not linear. It is quadratic, $\phi(x+y) = \phi(x) - \phi(y) = I(x, y) = x \cup y$. There are two such quadratic refinements and they are distinguished by $Arf \varphi \in \mathbb{Z}/2$. We get $\pi_2^{st} S^0 \cong \mathbb{Z}/2$, with $\Sigma \mapsto \Phi(\Sigma) = Arf \varphi$.

Question: In which dimensions is every element of $\pi_k^{st} S^0$ represented by a homotopy sphere?

Answer: It is true in all dimensions, except $2, 6, 14, 30, 62$, and possibly 126.

In the 60's Kervaire constructed an map

$$\varphi : H_{2k+1}(M^{4k+2}; \mathbb{Z}/2) \to \mathbb{Z}/2$$

which is a quadratic refinement of the intersection form, similar to the above. Here M is a very large class of manifolds which we will be a little vague about what structure is needed. The Kervaire invariant is $\Phi(M) = Arf \varphi$. Then if M is smooth, $\Phi(M) = 0$ if dim $M = 10$ or 18. Then X is two copies of the tangent bundle of the 5-sphere glued together, and $\partial X = N \cong S^9$ topologically, so $M = X \cup CN$. Then $\Phi(M) = 1$. Therefore M has *no smooth structure!*

Question: In which dimensions can $\Phi(M)$ be non-zero for smooth stably framed M?

Answer: Only in $2, 6, 14, 30, 62$, and possibly 126.

So far, this seems quite removed from homotopy theory. It was a fantastic triumph when Brouder in 1966 showed that $\Phi(M)$ is zero unless $\dim M = 2^{j+1} - 2$. It was believed that it was non-zero in all these dimensions. Moverover, he showed that a manifold M exists if and only if there is an element $\theta_j \in \pi_{2^{j+1}-2} S^0$ representing h_j^2 at the E^2-term of the Adams spectral sequence. This identified the Kervaire invariant problem with something that had been touched before.

We are going to show, using equivariant homotopy theory, that most of the time this element can't exist.

1.2 DATE: Sept 2, 2011

Last time we were overviewing the history of the Kervaire invariant one problem. We ended with Browder's theorem, which we'll review. If you want more, there will be some videos of talks linked to the course website.

Theorem 1.2.1 (Browder)**.** *Unless* $\dim M = 2^{j+1} - 2$ $\Phi(M) = 0$. *There exists* $M^{2^{j+1}-2}$ *with* $\Phi(M) = 1$ *if and only if there exists* $\theta_j \in \pi^{st}_{2^{j+1}-2} S^0$ *represented by* h_j^2 *in the* E_2*-page of the Adams spectral sequence.*

Theorem 1.2.2 (Hill-H.-Ravenel)**.** $\Phi(M) = 0$ *unless* $\dim M = 2, 6, 14, 30, 62$*, and possibly* 126*, i.e.* θ_j *does not exist for* $j \geq 7$.

Ingredients of the proof: We construct a multiplicative cohomology theory Ω and we prove the following things.

1. Detection Theorem: If θ_j exists then it has a non-zero image in $\tilde{\Omega}_{2^{j+1}-2}(S^0) = \pi_{2^{j+1}-2}\Omega$. This is like understanding a Ω-degree of the (stable) map $\theta_j : S^{N+2^{j+1}-2} \to S^N$. This is the one place where we really use the details of h_j^2 and the Adams spectral sequence.

2. Gap Theorem: $\pi_i \Omega = 0$ for $-4 < i < 0$.

3. Periodicity Theorem: Ω is periodic with period 256.

These together prove the main theorem.

We are going to introduce a general technique for constructing a large family of cohomology theories. There will all have some periodicity and they will also have this same gap.

Example 1.2.3. Real K-theory KO has 8-fold periodicity and there is the same gap between -4 and 0. ◇

The detection theorem is the least interesting and most computational. It is where you have to understand what h_j^2 is. But it is largely going through a list of already known results. It allows you to select the cohomology theory Ω. In some way this is the part we understand the least. Part of the problem is that we don't have a good geometric understanding of what the Kervaire invariant means geometrically. We will delay the discussion of the detection theorem until later in the course.

So what about this theory Ω? When you think of Ω you should think of an analogy to the situation for K, KO, and $K\mathbb{R}$. $K\mathbb{R}$ is an equivariant cohomology theory for C_2-actions. It is the K-theory of complex bundles with C_2 acting equivariantly by complex conjugation. Then Ω is like KO and the analog of $K\mathbb{R}$ $\Omega_{\mathbb{O}}$ which has a C_8-action. Ω is the C_8-fixed points. [[★★★ Here $\mathbb{O}$ is for octonions... why?!!]].

We will discuss ideas in Atiyah's paper "K-theory and reality". This is where he constructs $K\mathbb{R}$ and deduces the 8-fold periodicity of KO from this. Bott periodicity gives $\tilde{K}^0(X \wedge S^2) \cong K^0(X)$. The same proof implies $\tilde{K}\mathbb{R}^0(X \wedge S^{\rho_2}) \cong K\mathbb{R}^0(X)$ where the C_2-space S^{ρ_2} is the 1-point compactification of the regular representation. Then with a beautiful argument he deduces the 8-fold periodicity of KO from this twisted 2-fold periodicity of $K\mathbb{R}$.

By construction, $\Omega_{\mathbb{O}}(X \wedge S^{\rho_8}) \cong \Omega(X)$; we have a twisted 8-fold periodicity. Where ρ_8 is the 1-point compactification of the regular representation of C_8. This will imply that Ω has a 256 periodicity.

C_2 is the Galois group of $\mathbb{C}$ over $\mathbb{R}$. The C_8 is trying to be the Galois group of $\mathbb{O}$ over $\mathbb{R}$. Construct the action of C_8 on $\mathbb{O}$ with fixed points $\mathbb{R}$ and the Hilbert basis theorem holds as a C_8-representation, i.e. $\mathbb{O} = \rho_8$. You can't get this to work with algebra morphisms. However... Fact there is an anti-automorphism $\sigma : \mathbb{O} \to \mathbb{O}$, unique up to inner automorphism (conjugation). We don't know if there is a deeper connection between $\Omega_{\mathbb{O}}$ and $\mathbb{O}$.

A big tool we will use in this story is a way to decompose equivaraint cohomology theories into smaller pieces. This tool is the slice tower. This is the analog of the postnikov tower. Every space and every spectrum we have the Postnikov sections $P^nE \to P^{n-1}E$ with fiber an Eilenberg-MacLane spectrum $H\pi \wedge S^n$ where $\pi = \pi_n E$.

Let's now think about K-theory. $K \to \{P^n K\}$, the associated graded is a product $\prod_{z\in\mathbb{Z}} H\mathbb{Z} \wedge S^{2n}$. This trivially has the same 2-periodicity. Now remember that $K\mathbb{R}$ was periodic for S^{ρ^2}. We could build something trivial for this too, $\prod_{z\in\mathbb{Z}} H\mathbb{Z} \wedge S^{n\rho_2}$. Now it is true, there is a filtration of $K\mathbb{R}$ whose associated graded is this trivial theory. This filtration is really the slice tower and is an analog of the Postnikov tower. It exists for any of these equivariant cohomology theories.

1.3 Unstable Equivariant Homotopy Theory DATE: Sept 7, 2011

Adam's paper 'prerequisites for Carlsson's work' is a good reference for equivariant cohomology. In the 80's Carlsson proved the Segal conjecture using methods from equivariant homotopy theory. At the time there wasn't much known and there was some miss-information. This paper provides background.

Spaces will be a compactly generated weak Hausdorff space. Compactly generated means it is the colimit of its compact subspaces. Weak hausdorff means that the diagonal is closed (using the product in compactly gen spaces; this is why it is not just Hausdorff).

Example 1.3.1. [[★★★ pict]] X_n is the union of two lines glued along $(\infty, -\frac{1}{n}] \cup [\frac{1}{7}n, \infty)$. There is a map $X_n \to X_{n+1}$ and the colimit is *not* the line with two origins (the colimit is top), it is just the line. ◇

[[★★★ Why weak Hausdorff and not just Hausdorff?]]

Another possible notion of space is a simplicial set.

Let G be a finite group. Consider the category of spaces with a (left) G-action. We have to decide (or at least set up notation) for what the maps are. $\mathcal{T}^G$ has maps which are equivariant maps. $\mathcal{T}_G$ has maps all maps, or rather $\mathcal{T}_G(X,Y)$ is the G-space of maps. $\mathcal{T}^G(X,Y) = \mathcal{T}_G(X,Y)^G$ (fixed points).

We want the Homotopy Theory of G-spaces. There are two parts: abstract homotopy theory (setting up the model category) and then there are things which are unique to this theory. These we'll call computational aspects. The first thing people needed was a good class of spaces to work on. When this was coming into being it was already known that CW-complexes were a good class of spaces for ordinary homotopy theory.

G-CW complex should be something built from equivariant cells. We have to decide what is an equivariant cell.

Example 1.3.2. (1) The unit circle with $\mathbb{Z}/2$ acting by flipping. Another example: (2) S^1 with antipodal $\mathbb{Z}/2$-action. ◇

Classically a cell is a ball and we attach by gluing the boundary. Idea 1: A cell should be the unit ball in V where V is a representation of V with a G-invariant inner product. (no problem as G is finite). Then Example one is $D(V)/S(V)$ where $V = \sigma$ the sign representation. Every representation has a fixed point (the origin) and so we can't build the second example since it has no fixed points. So we need more then just representations.

Let $X_\bullet$ be a simplicial set with a G-action (or equivalently a simplical G-set). Then the geometric realization of this should be a G-CW complex. They are just too nice and combinatorial.

$$\begin{tikzcd} X'_n \times \partial\Delta^n \arrow[r, hook] \arrow[d] & X'_n \times \Delta^n \arrow[d] \arrow[dl, phantom, "\lrcorner"] \\ |sk^{n-1}X| \arrow[r] & |sk^n X| \end{tikzcd}$$

where X'_n is the G-set of non-degenerate simplices. This suggests that a G-cell should be a space of the form $S \times \Delta^n$ where S is a discrete (finite) G-set and G acts trivially on Δ^n.

Definition 1.3.3. A G-CW complex is a G-space built from G-cells of the form $S \times D^n$. ◇

In the literature people often say cells of the form $(G/H) \times D^n$. These are equivalent, but this involves decomposing S into orbits and choosing a point in each orbit. Now the first example has two 0-cells with the trivial action and a 1-cell which looks like $\mathbb{Z}/2 \times D^1$. The second antipodal example has a 0-cell $\mathbb{Z}/2 \times D^0$ and a 1-cell $\mathbb{Z}/2 \times D^1$.

So now $[X,Y]^G = \pi_0\mathcal{T}(X,Y)^G$ is homotopy class of equivariant maps. Whitehead's theorem says a weak equivalence $X \to Y$ of CW-complexes is a homotopy equivalence. We want a G-analog. The key issue is to understand what replaces the notion of homotopy groups. What we need is a condition guaranteeing that

$$[S \times S^{n-1}, X]^G \to [S \times S^{n-1}, Y]^G$$

is a bijection.

Suppose we know it is a bijection for $S_1 \times S^{n-1}$ and $S_2 \times S^{n-1}$, then it is so for $(S_1 \cup S_2) \times S^{n-1}$. So we might as well suppose that $S = G/H$. Now $\mathcal{T}^G(G/H \times A, B) = \mathcal{T}^H(A,B)$. In fact the restriction functor $\mathcal{T}^G \to \mathcal{T}^H$ has a left adjoint $G \times_H (-)$ and a right adjoint $\mathcal{T}^H(G,-)$ the space of H-equivariant maps of G into $(-)$. So...

$$[G/H \times S^{n-1}, X]^G = [S^{n-1}, X]^H = [S^{n-1}, X^H].$$

Theorem 1.3.4 (G-Whitehead Theorem)**.** *A map $X \to Y$ of G-CW complexes is an equivariant homotopy equivalence if and only if $[S^k, X^H] \to [S^k, Y^H]$ is a bijection.*

[[★★★ wait! bijections on unbased maps of sphere are not the same as bijection on homotopy groups. counter example: countble generic group (with single conj class) vs. uncountable generic group (with single conjugacy class).]]

Definition 1.3.5. A map $X \to Y$ of G-CW complexes is a weak equivalence if for all $H \subseteq G$, the maps $X^H \to Y^H$ is a weak equivalence. ◇

The G-equivariant homotopy theory of spaces is describable in terms of the ordinary homtopy theory of the H-fixed point spaces for all $H \subseteq G$.

example theorem: classical: X a CW-complex with $\dim X < n$ and Y a space such that $\pi_i Y = 0$ for $i \leq n$, then $[X, Y] = pt$.

The equivariant version, suppose that X is a G-CW complex. Then X^H is a CW complex of dimension $< n(H)$. Now suppose that Y is a G-space such that $\pi_i Y^H = 0$ for $i \leq n(H)$, then $[X, Y]^G = pt$.

The classical things like dimension, connectivity, etc. get changed to things which depend on a subgroup H of G. So really a homotopy group should have an index pair (i, H).

1.4 Slice Tower DATE: Sept 9, 2011

[[★★★ absent, will add notes later.]]

1.5 DATE: Sept 12, 2011

Our aim is to getting to stable equivariant homotopy theory. The unstable calculations are helpful, getting equianted with the techniques. One reason though that we are meandering is that I want to point out a decision that had to be made in the construction of the stable setting.

Stable homotopy via Spanier-Whitehead

You define a category where the objects are finite (pointed) CW-complexes and we have $\{X, Y\} = \lim_{n\to\infty}[S^n \wedge X, S^n \wedge Y]$. These are abelian groups.

- For a given X this limit is attained at a finite stage (independent of Y).
- In the stable range of dimensions, cofibrations and fibrations are the same thing. Cofibration sequences (in either X or Y) this gives a long exact sequence. This is the sense in which it is stable.
- There is Spanier-Whitehead duality: all objects are dualizable.
- Cohomology $H^*(-)$ is a functor on the Spanier-Whitehead category. This is because of the suspension isomorphism.

Equivariant Stable Homotopy

There is something called the G-Spanier-Whitehead category, where the objects are finite G-CW complexes. And the maps... well this is the place where there is a choice to make. There are really two viable approaches and they really have two different properties. It is a good idea to keep these choices in mind.

$$\{X, Y\}^G = \lim_{V\to\infty}[S^V \wedge X, S^V \wedge Y]^G$$

where V is a representation of G.

Example 1.5.1. How to organize the limit? Let U be a G-universe, i.e. a countably ∞-dimensional G-inner product space in which every irreducible representation occurs with infinite multiplicity. For example $U = \oplus^\infty \rho_G$. Then we can take the limit of all $V \subset U$, where V is G-stable and finite dimensional. If $V \subset W$ with compliment V' then

$$S^W \wedge X \cong S^{V'} \wedge S^V \wedge X \to S^{V'} \wedge S^V \wedge Y \cong S^W \wedge Y$$

◇

This is the official definition, but there are situations where we might run into a subset of these representations.

The alternate choice:

$$\{X, Y\}^G = \lim_{n \to \infty} [S^n \wedge X, S^n \wedge Y]^G$$

where the group does not act on the suspension coordinate.

Properties:

– For a given X the limit is obtained at a finite stage. This is true in the official and alternate versions.

– They are both stable: cofibration and fibration are the same. A cofibration in either variable gives long exact sequences.

– There is Spanier-Whitehead duality. Everything is dualizable in the official world, but this fails in the alternate world. (So no equivariant Poincaré duality in the alternate world.)

– Cohomology $H^*(-)$ is a functor on the Spanier-Whitehead category. This works in the alternate universe, but is more tricky (and sometimes fails) in the official universe.

In the last class we talked about $H^*(-; M)$ where $M : (fin^G)^{op} \to Ab$ taking $\sqcup$ to $\oplus$ and we wanted to know if this is a functor on the G-Spanier-Whitehead category?

$$H^*(Y; M) \cong H^{*+n}(S^n \wedge Y; M) \to H^{*+n}(S^n \wedge X; M) \cong H^*(X; M).$$

What about the official universe. This would work if we had:

$$H^*(Y; M) \cong H^{*+V}(S^V \wedge Y; M) \to H^{*+V}(S^V \wedge X; M) \cong H^*(X; M).$$

but then we have to make sense of these symbols. We run into the following problem.

Problem 1.5.2. Find a space $K(M, k + V)$ such that $\Omega^V K(M, k + V) = K(M, k)$. We want to de-loop be a representation.

Can we solve this problem? And the answer is not in general. For some M you can't even do this. For other M you can do it, but in multiple ways. There is extra data that needs to be specified.

The correct notion is that M must be a *Mackey functor*. We will explain these in a later lecture. We just wanted to point out some of the choices and decisions that needed to be made in setting up the stable theory of equivariant homotopy theory.

The Limit is obtained at a finite stage

We want to prove some things so this isn't just a day at the beach. But also there are some very important techniques which first appear in this calculation. The question: when is $[X,Y]^G \to [S^V \wedge X, S^V \wedge Y]^G$ an isomorphism?

Definition 1.5.3. $d(X)$: conj. classes of subgroups of $G \to \mathbb{Z}$. $d^H(X) = \dim X^H$. Connectivity $c(Y)$, if Y^H is $c^H(Y)-1$ connected. More generally, if $i : A \to X$, then $c^H(i) = c^H(X,A)$ is the smallest n such that $\pi_n(X^H, A^H) \neq 0$. $\diamond$

Let V be a G rep and let $S^{V_0} = (S^V)^H$. V_0 is the H-invariant part. Then $d^H(S^V) = \dim V_0$. and $c^H(S^V) = \dim V_0$. Then two lectures ago we showed that $[X,Y]^G_* = 0$ if $d(X) < c(Y)$ for all H. $\Omega^V Z = maps_*(S^V, Z)$ (non-equivariant maps with G acting by conjugation).

The map

$$[X,Y]^G \to [S^V \wedge X, S^V \wedge Y]$$

will be a bijection if $d(X) < c(Y \to \Omega^V S^V \wedge Y) - 1$. So we need to understand $c(Y \to \Omega^V S^V \wedge Y)$.

Remark 1.5.4. Non-equivariantly, $c(Y \to \Omega^n S^n \wedge Y) = 2c(Y)$ (by, for example, the Serre spectral sequence). $\diamond$

For a given group H we are looking at

$$Y^H \to (\Omega^V S^V \wedge Y)^H = map(S^V, S^V \wedge Y)^H$$

This later was studied, even independently of stable homotopy theory. For example in an ordianry algebraic topology course the antipodal $Z/2$-action calculation is the cornerstone calculation for the Borsuk-Ulam theorem. This was a very respectable calculation. There is a map

$$\begin{tikzcd} Y^H \arrow[r] \arrow[dr, "2c^H(Y)"'] & (\Omega^V S^V \wedge Y)^H \arrow[d, "?c"] \\ & \Omega^{V_0} S^{V_0} \wedge Y^H \end{tikzcd}$$

$V = V_0 \oplus W$, and we have $S^V = S^{V_0} \wedge S^W$. We also have a cofiber sequence

$$S(W)_+ \to S^0 \to S^W$$

cofiber sequence in pointed G-spaces, where $S(W)$ is the unit sphere in W. So, this gives us a cofiber sequence,

$$S^{V_0} \wedge S(W)_+ \to S^{V_0} \to S^V$$

and so now

$$\begin{tikzcd} & (\Omega^V S^V \wedge Y)^H \arrow[d, "?c"] & \\ Y^H \arrow[ur] \arrow[r, "2c^H(Y)"'] & \Omega^{V_0} S^{V_0} \wedge Y^H \arrow[r] & map(S^{V_0} \wedge S(W)_+, S^V \wedge Y) \end{tikzcd}$$

since the bottom need the connectivity of

$$map(S^{V_0} \wedge S(W)_+, S^V \wedge Y)$$

H-CW complex. Cells $H/K \times D^m$, $K \subseteq H$, K fixes a point of $S(W)$, $\dim V^K > \dim V^H$, $m \leq \dim V^K$. So the contribution from $H/K \times D^m$ will have the same connectivity as

$$map((H/K)_+ \wedge S^{V_0} \wedge S^m, S^V \wedge Y)^H = map(S^{V_0} \wedge S^m, S^V \wedge Y) = map(S^{V_0} \wedge S^m, S^{(V^K)} \wedge Y^H)$$

which has the connectivity of $\Omega^{V^K} S^K \wedge Y^K$ which is the connectivity of Y^K.

(by some kind of induction the K stuff is under control.)

1.6 DATE: Sept 14, 2011

Ambitious agenda today. We've introduced this equivariant Spanier-Whitehead category. The objects are G-CW complexes and

$$\{X, Y\} = \operatorname*{colim}_V [S^V \wedge X, S^V \wedge Y]^G$$

We showed last time that this stabilizes at some finite stage. What is $\{S^0, S^0\}^G$? In non-equivariant topology this is $\mathbb{Z}$, the index of a map.

To map a map of degree d, we choose a set S with d elements, embed it in a sphere S^n, Let BS be the collection of balls around these points. Then

$$S^n \to S^n/(S^n \smallsetminus BS) \simeq \vee_S S^n \to S^n$$

is a map of degree d.

There is an equivariant anaolg. S a finite G-set, $S \subset S^V$, and now we form,

$$S^V \to S^V/(S^V \smallsetminus BS) \simeq \vee S^V \to S^V$$

is a map of degree 'S'.

Example 1.6.1. $G = \mathbb{Z}/2$ and S has two elments and free $\mathbb{Z}/2$-action. Then $S \subset S^{\text{sign}}$, at north and south pole. [picture] ⋄

More generally, suppose we have a map of G-sets $p : S' \to S$. Then we can choose $S' \subset S \times V$ for some representation, and we can form the Pontryagin Thom collapse

$$S \times V/(S \times V \smallsetminus B) \to S \times V/S \times V \smallsetminus BS' \to V/(V \smallsetminus BS')$$

In stable homotopy we tend to avoid the notion $\times$ because it could me the product of the spaces, or it could mean the categorical product. These are different in general. In stead we'll write $X_+ \wedge Y_+$. Then the above can be rewritten as:

$$p^! : S_+ \wedge S^V \to S'_+ \wedge S^V$$

so we get a map going the other way.

The map from before was obtained by letting $S = pt$, and then we get $S^0 \to S'_+ \to S^0$, where the last map is induced from $S' \to pt$.

Proposition 1.6.2. *If*

$$\begin{tikzcd} T' \arrow[r, "j"] \arrow[d, "q"'] \arrow[dr, phantom, "\ulcorner", very near start] & S' \arrow[d, "p"] \\ T \arrow[r, "i"'] & S \end{tikzcd}$$

is a pullback diagram, in G-sets, then

$$\begin{tikzcd} T'_+ \arrow[r, "j"] \arrow[dr, phantom, "\ulcorner", very near start] & S'_+ \arrow[d, "p^!"] \\ T_+ \arrow[u, "q^!"] \arrow[r, "i"'] & S_+ \end{tikzcd}$$

commutes.

Burnside Category of G

The objects are finite G-sets and the morphisms from S to T is the free abelian group on the set of equivalences classes of diagrams $S \leftarrow S' \rightarrow T$, modulo the equivalence relation that disjoint unions of spans are sums in the group. Two spans are equivalent if there is an isomorphism of spans. To compose, we form the pullback.

Proposition 1.6.3. *The construction described at the start of class gives a functor from $Burn_G \to G$-Spanier-Whitehead category, sending S to S_+.*

The only real thing to prove is that the composition laws are compatible, but that is the substance of the previous proposition.

Theorem 1.6.4. *The functor $Burn_G \to G - SW$ is fully-faithful, i.e. $Burn_G(S,T) \cong \{S_+, T_+\}$.*

Example 1.6.5. $S = T = pt$, then $Burn_G(pt, pt) = \mathbb{Z}\{\text{finite } G - \text{sets}\}/(\sqcup \sim +) = A(G)$, the Burnside ring of G. ◇

Corollary 1.6.6. $\{S^0, S^0\}^G = A(G)$.

Remark 1.6.7. Note that $Burn_G(S,T) \cong Burn_G(T,S)$ abstractly. ◇

$A(G)$ is the free abelian group on the set $\{G/H \mid H \subset G$, representative of conjugacy class of finite subgroup There is a map $A(G) \to \mathbb{Z}$ sending S to $|S^H|$ for each conjugacy class of subgroups. Then we get an injective ring homomorphism

$$A(G) \to \prod_{\text{conj class} H \subset G} \mathbb{Z}.$$

This is an isomorphism after inverting the order of G.

Example 1.6.8. $G = \mathbb{Z}/2$, $A(G) \cong \mathbb{Z} \oplus \mathbb{Z}$. The map $A(G) \to \mathbb{Z} \times \mathbb{Z}$ maps S to $(|S|, |S^G|)$. So $A(\mathbb{Z}/2) = \{(x,y) \mid x \equiv y (\text{mod } 2)$. ◇

Spanier-Whitehead duality

$S \times S \leftarrow S \to pt$ gives us $S_+ \wedge S_+ \to S^0$, and the other map $pt \leftarrow S \to S \times S$ gives a map $S^0 \to S_+ \wedge S_+$. The claim is that this makes S_+ self dual.

Definition 1.6.9. X and Y are SW duals if there are maps $\mu : X \wedge Y \to S^0$ and $\eta : S^0 \to Y \wedge X$ such that the zig-zag equations are satisfied:

$$(1 \wedge \mu) \circ (\eta \wedge 1) = 1$$
$$(\mu \wedge 1) \circ (1 \wedge \eta) = 1$$

(It is enough just to have these composites be isomorphisms.) ◇

Exercise 1.6.10. In this case, $\{A \wedge X, B\} \cong \{A \wedge B \wedge Y\}$.

Exercise 1.6.11. These maps $S_+ \wedge S_+ \to S^0$ and $S^0 \to S_+ \wedge S_+$ give an SW duality of S_+ with itself.

Thus we have $\{S_+, T_+\} \cong \{S^0, S_+ \wedge T_+\} \cong \{T_+, S_+\}$.

There are three proofs of the main theorem. One is to prove that you have enough transversailty and to use framed equivariant cobordisms. There is also a proof using Segal's Γ-spaces. There is also Tom Dieck's book which probably uses cobordism.

proof of the main theorem. Induction on the order of G. Suppose we know the theorem for all proper sub groups of G. Then $\{G/H_+, S^0\}^G = \{S^0, S^0\}^H \cong \{S^0, G/H_+\}^G$ by duality. So we know the result for $\{S, S'\}$ if one of S, S' has no elements fixed by all of G. So we are reduced the assertion that $A(G) \to \{S^0, S^0\}^G$ is an isomorphism.

Let ρ be the regular representation, $\rho = 1 \oplus \overline{\rho}$, where $\overline{\rho}$ is ht reduced regular representation. Then $S(\overline{\rho})$ is the unit sphere, and $S^{\overline{\rho}}$ is the one-point compactification. There is a cofibration sequence:

$$S(\overline{\rho})_+ \to S^0 \to S^{\overline{\rho}}.$$

Get a long exact sequence, $\pi_n^G(X) = \{S^n, X\}^G$.

[bell tolls]. □

1.7 DATE: Sept 16, 2011

Where were we? $Burn_G$ is the burnside category of a finite group. The objects are finite G-sets. The maps from S to T are the group completion of spans modulo a certain equivalence relation. We constructed a functor

$$Burn_G \to G - SW$$

into the Spanier-Whitehead category, which sends $S \times T$ (note: NOT the categorical product!) to $S_+ \wedge T_+$. It is symmetric monoidal. This implied that S_+ is self-dual, $\{S_+ \wedge X, Y\}^G \cong \{X, S_+ \wedge Y\}^G$.

Theorem 1.7.1. *The functor $Burn_G \to G{-}SW$ is fully-faithful, $Burn_G(S,T) \cong \{S_+, T_+\}^G$.*

Proof. Induction on $|G|$. The induction hypothesis implies that we know the result for pairs S and T provided one of S or T has no G-fixed points. This is because then

$$\{S_+, T_+\}^G = \{S_+ \wedge T_+, S^0\}^G = \oplus_\alpha \{(G/H)_+, S^0\}^G = \oplus \{S^0, S^0\}^{H_\alpha}.$$

Recall that $A(G)$ = Burnside ring of G = group completion of the monoid of finite G-sets. The induction hypothesis reduces us to showing $A(G) \to \{S^0, S^0\}^G$ is an isomorphism.

There is another consequence of the induction hypothesis. Let X be a (finite) pointed G-CW complex with no fixed points other then the base point. There is the skeletal filtration.

$$\begin{tikzcd} \vee(G/H)_+ \wedge S^0 \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \vee(G/H_+ \wedge D^1) \arrow[d] \\ X^{(0)} \arrow[r] & X^{(1)} \end{tikzcd}$$

(etc). This gives a long exact sequence in stable homotopy,

$$\{S^0, \vee(G/H)_+ \wedge S^0\}^G \to \{S^0, X^{(0)}\}^G \to \{S^0, X^{(1)}\}^G \to \{S^0, \vee(G/H)_+ \wedge S^1\}^G = 0$$

and the second term is $Burn_G(pt, \overline{X}_0)$ where $X^{(0)} = \overline{X}_0 \sqcup pt$. This last group is zero because $\{S^0, S^1\}^H = 0$. The conclusion is thus,

$$Burn_G(pt, \overline{X}_0) \to \{S^0, X\}^G$$

is surjective.

We are going to see this kind of argument over and over again. Study it! Once you've got used to the yoga of equivariant homotopy theory, this feels like an easy proof. Here is the great trick. We'll formulate it systematically later on, for now it will be ad hoc. Let ρ be the regular representation and $\overline{\rho}$ the reduced regular representation ($\dim = |G|-1$). Now $S(\overline{\rho})$ is the unit sphere and $S^{\overline{\rho}}$ is the one point compactification. So there is a cofibration sequence,

$$S(\overline{\rho}) \to S^0 \to S^{\overline{\rho}}.$$

We will use this cofibration sequence. Now $S(\overline{\rho})$ has no fixed points, so in principle we understand it by induction. Miraculously $S^{\overline{\rho}}$ can also be understood. Let $S = (S(\overline{\rho}))^{(0)}$.

$$\begin{tikzcd} Burn_G(pt, S) \arrow[r] \arrow[d, two heads] & A(G) \arrow[r] \arrow[d] & {} \arrow[d] \\ \{S^0, S(\overline{\rho})_+\}^G \arrow[r] & \{S^0, S^0\} \arrow[r] & \{S^0, S^{\overline{\rho}}\} \end{tikzcd}$$

Image of $A(G)$ in $\{S^0, S^0\}^G$ contains the image of $\{S^0, S(\overline{\rho})_+\}^G$.

Now $\{S^0, S^{\overline{\rho}}\}^G = \operatorname{colim}[S^V, S^{V+\overline{\rho}}]^G$ which can be computed for a large representation V. Let $V_0 \subset V$ be the G-invariant elements. Then

$$[S^V, S^{V+\overline{\rho}}]^G \to [S^{V_0}, S^{V+\overline{\rho}}]^G = [S^{V_0}, S^{V_0}]^G = \mathbb{Z}.$$

Claim: This map is an isomorphism.

Proof. Suppose that we are midway in an equivaraint cell decomposition from S^{V_0} to S^V.

$$\begin{tikzcd} & \vee(G/H)_+ \wedge S^{m-1} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \vee(G/H_+ \wedge D^m) \arrow[d] & \\ S^{V_0} \arrow[r, dashed] & (S^V)^{(m-1)} \arrow[r] & (S^V)^{(m)} \arrow[r, dashed] & S^V \end{tikzcd}$$

Now all the cells we are adding must have proper H. For a given H, $m \leq \dim V^H$. So only certain m and H occur.

$$\prod[(G/H)_+ \wedge S^m, S^{V+\overline{\rho}}]^G \to [(S^V)^{(m)}, S^{V+\overline{\rho}}] \to [(S^V)^{(m-1)}, S^{V+\overline{\rho}}] \to \prod[(G/H)_+ \wedge S^{m-1}, S^{V+\overline{\rho}}]^G$$

Now

$$[(G/H)_+ \wedge S^m, S^{V+\overline{\rho}}]^G = [S^m, S^{V+\overline{\rho}}]^H = [S^m, (S^{V+\overline{\rho}})^H].$$

$\dim(S^{V+\overline{\rho}})^H = \dim V^H + \dim \overline{\rho}^H > m$. So this is a map of a sphere into a bigger sphere, so this group is zero. □

So now we have,

$$\begin{tikzcd} Burn_G(pt, S) \arrow[r] \arrow[d, two heads] & A(G) \arrow[d] \arrow[dr] & \\ \{S^0, S(\overline{\rho})_+\}^G \arrow[r] & \{S^0, S^0\} \arrow[r] & \{S^0, S^{\overline{\rho}}\} = \mathbb{Z} \end{tikzcd}$$

Now $A(G)$ is the free abelian group on $\{G/H \mid H \text{ is a rep of a conj class of subgorup}\}$. There is another way to get maps to the integers.

$$A(G) \hookrightarrow \prod_{H \subset G} \mathbb{Z}$$

where each map sends $S \mapsto |S^H|$, and the product is over conjugacy classes. This is a ring homomorphism and is an iso after inverting $|G|$.

Exercise 1.7.2. These maps to $\mathbb{Z}$ factor as $A(G) \to \{S^0, S^0\} \to \mathbb{Z}$, where the last map sends $[S^V, S^V]^G \to [S^{V^H}, S^{V^H}]$. Hence $A(G) \to \{S^0, S^0\}$ is injective.

□

The Segal conjecture is that $[BG_+, S^0] = A(G)\hat{}_I$ (completion at the augmentation ideal). People has solved this for cyclic groups, but no one had been able to get the general group case. In the 80's gunnar carlson used this method of *isotropy separation* to prove the Segal conjecture for general groups.

1.8 DATE: Sept 19

No class on Wednesday. There are some problem sets on the course website.

SW_G is the spanier whitehead category. Bun_G is the Burnside category. $Burn_G \to SW_G$ sending S to S_+.

Theorem 1.8.1. *The above is fully faithful.*

In otherwords, the map from $Burn_G(S,T) \to \{S_+, T_+\}^G$ is an isomorphism.

Equivariant homotopy groups $[S^n, X]^G$ is part of a contavariant additive functor from finite G-sets to abelian groups.

$$\underline{\pi}_n X(S) = [S^n \wedge S_+, X]$$

These functors are called *coefficient systems*. Let $\underline{\pi}_n^{st}(X)(S) = \{S^n \wedge S_+, X\}^G$. This is a contravariant additive functor

$$Burn_G \to Ab$$

This is called a *Mackey Functor*. We showed before that every coefficient system occurs as a homotopy group, and we were even able to form an Eilenberg-MacLane space $K(A,n)$ for A a coefficient system.

An alternative definition of a Mackey Functor. Set_G = the category of finite G-sets. A Mackey functor is a pair of functors

$$M_* : Set_G \to Ab M^* : \qquad (Set_G)^{\mathrm{op}} \to Ab$$

such that $M_*(S) = M^*(S) = M(S)$, and for every pullback square

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "p"'] \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[d, "q"] \\ S \arrow[r, "j"'] & T \end{tikzcd}$$

the induced square commutes:

$$\begin{tikzcd} M(A) \arrow[r, "i_*"] & M(B) \\ M(S) \arrow[r, "j_*"'] \arrow[u, "p^*"] & M(T) \arrow[u, "q^*"'] \end{tikzcd}$$

We'll show that every Mackey functor occurs as a stable homotopy group and in M is a Mackey functor and we'll show that for every representation V, $K(M,n) \simeq \Omega^V K(M, n+V)$ where this later space is to be constructed. In other words we can deloop be any representation.

We won't quite get a space X for which $\underline{\pi}_n^{st} X = M$, what we will get instead is a space X such that

$$S \mapsto [S^W \wedge S_+, X]$$

is M for a large W. This are the 0-homotopy groups of $\Omega^W X$, but these are not the same as stable 0-homotopy groups. Eventually we will see that there aren't enough objects in SW_G.

Suppose M is a Mackey functor. **Step 1** Choose W a large representation such that $[S^W, S^W]^G \cong \{S^0, S^0\}^G$, that is for all V, $[S^W, S^W]^G \to [S^W \wedge S^V, S^W \wedge S^V]$ is a bijection. For this W,

$$[S_+ \wedge S^W, T_+ \wedge S^W]^G \cong \{S_+, T_+\}^G = Burn_G(S,T)$$

step 2 We can get any representatble functor be $\{-, T_+\}^G$. An arbitrary sum of representable Mackey functors P occurs as $\{-, T\}^G$ where T is a possibly infinite discrete pointed G-set, i.e. $\underline{\pi}_0^{st,G} T_+ = P$.

step 3 Suppose M is arbitrary, then there exists

$$P \to Q \twoheadrightarrow M$$

where P and Q are sums of representable functors.

We can find a map $S^W \wedge T_+ \to S^W \wedge U_+$ and we can form the mapping cone X_1. Then we get an exact sequence of Mackey functors,

$$[S^W \wedge S_+, S^W \wedge T_+]^G \to [S^W \wedge S_+, S^W \wedge U_+]^G \to [S^W \wedge S_+, X_1]^G \to [S^W \wedge S_+, \Sigma S^W \wedge T_+]^G$$

Claim: For any representation V, $[S^V, S^1 \wedge S^V]^G = 0$.

Proof. $d^H(S^V) = \dim V^H$, $c^H(S^1 \wedge S^V) = 1 + \dim V^H$. □

Our sequence becomes

$$P \to Q \to [S^W \wedge S_+, X_1]^G \to 0$$

So that $[S^W \wedge S_+, X_1]^G = M$. By forming the cone over all maps $S^1 \wedge S^W \wedge T_+ \to X_1$ we can build X_2 such that $[S^1 \wedge S^W \wedge S_+, X_2]^G = 0$. This leads to $K(M, W)$ such that

$$[S^n \wedge S^W \wedge S_+, K(M, W)] = M \text{ if } n = 0 \quad 0 \text{ else.}$$

Using this we can form $K(M, W + V + n)$, and define $K(M, V + n) = \Omega^W K(M, W + V + n)$. Conclusion: $\Omega^V K(M, V + n) = K(M, n)$.

Non-equivariantly, the Eilenberg Maclane space $K(A, n)$ is characterized by $\pi_n = A$, $\pi_i = 0$ for $i \neq n$. You would like to say that $K(M, W)$ is characterized by

- $[S^i \wedge S^W \wedge S_+, K(M, W)] = 0$ when $i \neq 0$ and is M when $i = 0$, and
- $K(M, W)$ is built from cells of the form $S^W \wedge S_+ \wedge D^n$.

Part of the problem is that the first one is like saying the $\pi_{W+i} = 0$ for $i > 0$. But we can't really talk about lower negative i.

Problems with SW^G

1. For $f \in \{X, Y\}^G$, we can't form $Y \cup CX$. We only get the map after smashing with S^W for sufficiently large W. all we can form is $S^W \wedge Y \cup CX$. We would like to define $Y \cup CX$ as $S^{-W} \wedge (\text{cofibr} S^W \wedge Y \to S^W \wedge X)$.

We could add objects $S^{-W} \wedge X$ by defining $\{Y, S^{-W} \wedge X\}^G \cong \{S^W \wedge Y, X\}^G$.

We also need to add colimits. But we could have sequences

$$X_0 \to S^{-W_1} \wedge X_1 \to S^{-W_2} \wedge X_2 \cdots$$

We could then define maps between these objects (since they are colimits).

The (homotopy) category of G-CW spectra is the category obtains from SW^G by adding $S^{-W} \wedge X$ and colimits. In the end it all comes back to these kinds of calcualtions with SW_G.

1.9 DATE: Sept 23, 2011

SW_G the Spanier-Whitehead category. Inadequacies

– $S^{-V} \wedge X$ is missing.

– colimits are missing

We will construct equivariant spectra by adding these objects $S^{-V} \wedge X$ and colimits. To do this in a clean way, we will need to add some more machinery. First we will just start with the naive approach and then slowly add more gadgets.

Approach 1: We want object $S^{-V} \wedge X$, and we also want colimits, for example filtered colimits $\operatorname{colim}_n S^{V_n} \wedge X_n$, where the index n ranges over some represenations. This is equivalent to notation $\operatorname{colim}_V S^V \wedge X_V$. An arbitrary system like this gives you a collection of spaces X_V and a stable map in $\{S^W \wedge X_V, X_{V\oplus W}\}^G$. It is a little inconvenient to have stable maps at this point, so it is a little nicer to look at non-stable maps. It suffices to form a more specialized system where you have a collection X_V and an *unstable* map $S^W \wedge X_V \to X_{V\oplus W}$. We get a definition which doesn't need to leave the realm of unstable equivariant theory:

Definition 1.9.1 (tenetitive)**.** A *G-spectrum* is a colloection of G-spaces $\{X_V$ — V a rep of G $\}$ together with maps $S^W \wedge X_V \to X_{V\oplus W}$ satisfying some associativity. ◇

Imagine that $X = \{X_V\}$ is the colimit of $S^{-V} \wedge X_V$. There are a couple of problems with this. One is that there are too many Vs we need to somehow restrict to a smaller class of these V.

Remedy (May): We let U be a *G-universe* a countably infinite dimensional G-inner product space, containing every fintie dimensional representation of G infinitely often. For example $U = \oplus_\infty \rho$ infinitely many copies of the permutation representation. If $V \subset W \subset U$, then $W - V =$ orthogonal compliment of V in W.

Definition 1.9.2. A *G-spectrum indexed on U* is a collection of spaces $\{X_V\}$ where $V \subseteq U$ is a finite G-subspace of U together with maps $S^{W-V} \wedge X_V \to X_W$, which are associative in the evident sense. ◇

There are lots of variations on this. For example may we just take a chain of subspaces V.

The other approach is what is called **Orthogonal Spectra**. We assume all our representations come with an inner product. $O(V,W)$ is the Stiefel-manifold of non-equivariant inner product preserving maps $V \to W$. The group G acts on $O(V,W)$ diagonally. A G-fixed point is an equivariant embedding.

Write $W - V$ for the vector bundle over $O(V,W)$ whose fiber over $f : V \hookrightarrow W$ is the orthogonal compliment of $f(V)$. Let $I_G(V,W)$ be the Thom-complex $Th(O(V,W), W-V)$ $= (W-V)_\infty$. This is a family of $S^{W-f(V)}$. This is a G-space.

Definition 1.9.3. An *orthogonal G-spectrum X* is a collection of spaces X_V for every orthogonal G-representation, together with G-equivariant maps $I_G(V,W) \wedge X_V \to X_W$, which is associative in the obvious sense. ◇

Really this is defining a functor from a category to G-spaces.

Exercise 1.9.4. Describe an orthogonal spectrum as a functor from a category I_G to G-spaces (and non-equivariant maps). Really these are enriched over G-spaces and equivariant maps.

There is an obvious notion of map in both of these approaches. For example in the universe style, such an obvious map $X \to Y$ is just a collection of maps $X_V \to Y_V$ making the diagrams commute. This is a little bit wrong.

Example 1.9.5. For any G-space T, we can form a spectrum $\Sigma^\infty T$ where $(\Sigma^\infty T)_V = S^V \wedge T$. This is supposed to give the embedding of our category SW_G into spectra. $\Sigma^\infty T \sim \operatorname{colim} S^{-V} \wedge S^V \wedge T = T$. What are the naive maps $\Sigma^\infty S^0 \to \Sigma^\infty S^0$? Well we get a map $f : S^0 \to S^0$ by looking at zeroth space, and the compatibility implies that this determines all the maps. Hence we just get space maps $S^0 \to S^0$, but not Stable maps! ◇

Fix a representation U. The define $(S)_W = *$ if $U \not\subseteq W$ and S^W if $U \subseteq W$. This object is supposed to correspond to the object $\operatorname{colim} S^{-W} \wedge S_W = \operatorname{colim}_{U \subset W} S^{-W} \wedge S^W = S^0$. Spectrum maps from this S into S^0 are the same as G-space maps $S^U \to S^U$. For U sufficiently large this does give $\{S^0, S^0\}$. So we need to introduce a class of weak equivalence so that this operation doesn't change the spectrum. This is where we really need the model category structure.

1.10 Model Categories DATE: Sept 26, 2011

Last time we introduced the category of orthogonal spectra and there were two things that came up. One was that to get the correct notion of map we had to replace one object with another. A *model category* in the sense of Quillen is a category $\mathcal{C}$ which is closed under all small limits and colimits, a equipped with the following structure: three classes of maps called cofibrations ($\hookrightarrow$), fibrations ($\twoheadrightarrow$), and weak equivalences ($\simeq$), satisfying the following properties:

M1 retracts of cofibrations, weak equiv, or fibration are the same.

M2 If two maps in a composition is a weak equivalence, then so is the third.

M3 Every map $\to$ admits factorizations

$$\begin{tikzcd} {} \arrow[r, hook] \arrow[d, hook, "\simeq"'] \arrow[dr] & {} \arrow[d, two heads, "\simeq"] \\ {} \arrow[r, two heads] & {} \end{tikzcd}$$

M4 Lifting property:

$$\begin{tikzcd} {} \arrow[r] \arrow[d, hook, "\simeq"'] & {} \arrow[d, two heads] \\ {} \arrow[r] \arrow[ur, dashed, "\exists"] & {} \end{tikzcd}$$

$$\begin{tikzcd} {} \arrow[r] \arrow[d, hook] & {} \arrow[d, two heads, "\simeq"] \\ {} \arrow[r] \arrow[ur, dashed, "\exists"] & {} \end{tikzcd}$$

– [[★★★ Closure under base change?]]

A model category $\mathcal{C}$ which is pointed ($\emptyset \simeq pt$) has functors $\Sigma : h\mathcal{C} \leftrightarrow h\mathcal{C} : \Omega$. A pointed model category is *stable* if these are equivalences. This implies that cofibration sequences and fibration sequences are the same.

Example 1.10.1. We have G-spaces τ^G. The category is based spaces with a G-action, and the maps are equivariant maps. A map $X \to Y$ is a weak equivalence if for all subgroups $H \subset G$, the map of fixed point spaces $X^H \to Y^H$ is a weak equivalence. A map $X \to Y$ is a fibration if $X^H \to Y^H$ is a Serre fibration for all H. In the proof that this is a model category, you learn that the cofibrations are retracts of the *cellular maps* which are maps obtained by attacking cells of the form:

$$(G/H)_+ \wedge S_+^{n-1} \hookrightarrow (G/H)_+ \wedge D_+^n.$$

Maps in the homotopy category between G-CW complexes are just G-equivariant homotopy classes of maps. ◇

$\mathcal{S}^G$ is the category of equivariant orthogonal spectra. The objects are collections of spaces X_V and maps $I_G(V,W) \wedge X_V \to X_W$, satisfying the obvious associativity constraint. A map is a collection of maps $X_V \to Y_V$ such that for all V, W the following diagram commutes,

$$\begin{tikzcd} I_G(V,W) \wedge X_V \arrow[r] \arrow[d] & X_W \arrow[d] \\ I_G(V,W) \wedge Y_V \arrow[r] & Y_W \end{tikzcd}$$

We saw last time these maps were a little bit inadequate. The model category structure will rectify this. We will also reformulate this as a functor category later on (when we talk about smash product). Suppose that $X \in \mathcal{S}^G$, the stable homotopy groups $\pi_n^H X$ for $H \subseteq G$ and $n \in \mathbb{Z}$ are defined as

$$\operatorname*{colim}_{V \to \infty} [S^{n+V}, X_V]^H.$$

Remark 1.10.2. For $V >> 0$ means it contains at least $-n$ trivial representations, $V = \mathbb{R}^{-n} \oplus V'$, and then $S^{n+V} \cong S^{V'}$. ◇

Remark 1.10.3. Call a sequence $V_1 \subset V_2 \subset \cdots$ of reps *exhausting* if every rep W of G embedds in V_n for $n >> 0$. Let $\iota_n \in O(V_n, V_{n+1})$, then $S^{V_{n+1}-V_n} \subset I_G(V_n, V_{n+1})$. So our map $I_G(V_n, V_{n+1}) \wedge X_{V_n} \to X_{V_{n+1}}$ gives $S^{V_{n+1}-V_n} \wedge X_{V_n} \to X_{V_{n+1}}$. Using just these maps, we may calculated π_n^H as this filtered colimit. The key point is the connectivity of Steiffel manifolds. ◇

More generally, let's define a *coefficient system* $\underline{\pi}_n X$,

$$\underline{\pi}_n X(S) = \operatorname*{colim}_{V \to \infty} [S^n \wedge S^V \wedge S_+, X_V]^G.$$

Then

1. $\underline{\pi}_n X(G/H) = \pi_n^H X$.
2. Equivariant stability implies that $\underline{\pi}_n X$ has the structure of a Mackey functor.

We make the weak equivalence the stable equivalences, those maps that induce isomorphisms on $\underline{\pi}_n$. But to do this and define the cofibrations and fibrations we need to introduces some auxiliary model structures.

Definition 1.10.4. A map $X \to Y$ in $\mathcal{S}^G$ is a *level equivalence* if for all V, $X_V \to Y_V$ is a weak equivalence in τ^G. ◇

Level equivalence implies stable equivalence, but not the other way around.

Example 1.10.5. T a pointed G-space, $(\Sigma^\infty T)_V = S^V \wedge T$. Fix m, then $\Sigma^{\infty\prime} = S^V \wedge T$ if $\dim V > m$ and is *pt* otherwise. $\Sigma^{\infty\prime} T \to \Sigma^\infty T$ is a stable equivalence, but not a level equivalence. ◇

There is the *level model structure*

- $X \to Y$ is a fibration if $X_V \to Y_V$ is a fibration in τ^G for all V.
- the weak equivalences are the level equivalences.

(This is like the projective model structure, but where the diagram category is enriched in G-spaces).

There is the *positive level model structure*

- A map $X \to Y$ is a positive level fibration (respectively weak equivalence) if for all V with $\dim^G V > 0$ $X_V \to Y_V$ is a fibration (resp, weak equivalence).

This was a very good idea of Jeff Smith which comes in when we want to discuss ring spectra. It will become clear later why we need this.

The *stable model structure* is obtained from the positive level model structure by localizing at the stable equivalences. We will say more carefully what we mean by localizing in the next class.

1.11 DATE: Sept 28, 2011

Equivariant Orthongal Spectra. There are a lot of things to say. We could spend the rest of the course on the homotopy theory of equivariant orthogonal spectra, but we have other goals. But we will spend a few lectures discussing them. We are using this to set up the following.

- (equivariant) stable model category $\mathcal{S}^G$,

– which is tensored and cotensored over G-spaces τ^G. Given $E \in \mathcal{S}^G$ and $X \in \tau^G$, we get $E \wedge X \in \mathcal{S}^G$ and $E^X \in \mathcal{S}^G$, which are adjoints.

  Stable means that Σ and Ω are inverses up to weak equivalence. *Equivariant* stable means that $E \mapsto E \wedge S^V$ and $E \mapsto E^{S^V} = \Omega^V E$ are inverse up to weak equivalence. (The units and counits are weak equivalences).

– Up to this point, the trivial model category 1 gives this. So we require more, there should be a Quillen pair: $\Sigma^\infty : \tau^G \leftrightarrows \mathcal{S}^G : \Omega^\infty$, and for X, Y finite pointed G-CW complexes, $h\mathcal{S}^G(\Sigma^\infty X, \Sigma^\infty Y) = \{X, Y\}^G$.

  There are a lot of categories that satisfy these. We saw two, one given by choosing universes and one by equivariant orthogonal spectra

– A symmetric monoidal structure $\wedge \mathcal{S}^G \times \mathcal{S}^G \to \mathcal{S}^G$ compatible with the things we've written so far, $E \wedge X \simeq E \wedge \Sigma^\infty X$. The unit is S^0. This implies that $\Sigma^\infty X \simeq S^0 \wedge X$.

We have the definition; we want to prove (maybe in the following lectures):

1. equivariant stability,

2. the relation with the equivariant Spanier-Whitehead category.

A category I_G. The objects are finite dimensional orthogonal G-reps V and $I_G(V, W) = Thom(O(V, W), V - W)$ (maps from $V \to W$) is a G-space and not a space. I_G is enriched over G-spaces. We have

$$I_G(V, W) \wedge I_G(U, V) \to I_G(U, W)$$

lying over $O(V, W) \times O(U, V) \to O(U, W)$. This is an equivariant map, so we are enriched over τ^G.

The category τ_G is also enriched over τ^G.

Definition 1.11.1. An equivariant orthogonal spectrum is a functor $I_G \to \tau_G$ of categories enriched over τ^G. ◇

What is an enriched functor? It associated to every V space X_V and for every V, W a map $I_G(V, W) \to \tau_G(X_V, X_W)$ an equivariant map of G-spaces. And also an associativity condition for every triple of representations. Equivalently, this is the same data as

$$I_G(V, W) \wedge X_V \to X_W$$

equivariant. So $\mathcal{S}^G = \tau_G^{I_G}$ is a diagram category. This highlights certain special functors, the representable functors. Given V, we have $I_G(V, -)$. We will call this object $S^{-V} := I_G(V, -)$. By Yoneda, $\mathcal{S}^G(S^{-V}, E) = E_V$. The category $\mathcal{S}^G$ is tensored over τ^G. $(E \wedge X)_V = E_V \wedge X$. In particular, if $A \in \tau^G$, then maps $S^{-V} \wedge A \to E$ are in 1-1 correspondence with equivariant maps $A \to E_V$.

Let's go back to the model structure on τ^G. A map $X \to Y$ is a fibration/weak equivalence if for all $H \subseteq G$, $X^H \to Y^H$ is a Serre fibration/weak equivalence. $X \to Y$ is a fibration if and only if it has the right lifting property with respect to the maps $G/H \times I^{n-1} \to (G/H) \times I^n$. It is an acyclic fibration iff it has the right lifting property with respect to $G/H \times S^{n-1} \to G/H \times D^n$.

The Level model structure on $\mathcal{S}^G$ has fibrations $X \to Y$ iff $X_V \to Y_V$ is a fibration for all V, and similarly for weakequivalence. Being a level fibration in $\mathcal{S}^G$ is equivalent to having the right lifting property for

$$S^{-V} \wedge (G/H \times I^{n-1})_+ \to S^{-V} \wedge (G/H \times I^n)_+.$$

It is similar for level acyclic fibrations. In otherwords,

$$\mathcal{A} = \left\{ S^{-V}(G/H \times I^{n-1})_+ \to S^{-V} \wedge (G/H \times I^n)_+ \right\}$$
$$\mathcal{B} = \left\{ S^{-V}(G/H \times S^{n-1})_+ \to S^{-V} \wedge (G/H \times D^n)_+ \right\}$$

Are the generating level acyclic cofibrations and cofibrations.

1.12 DATE: Sept 30, 2011

Last time we discussed $\mathcal{S}^G$ as an enriched functor category, $I_G \to \tau_G$. This highlighted S^{-V}, the object corepresented by V. For the next couple lectures we will focus on category theoretical aspects, then pick up homotopy theory after. There is the *canonical presentation*. Suppose we have a spectrum X. What does it mean to give a map $X \to Y$? It means to give a collection of maps $X_V \to Y_V$ such that for all V, W,

$$\begin{array}{ccc} I_G(V,W) \wedge X_V & \longrightarrow & X_W \\ \downarrow & & \downarrow \\ I_G(V,W) \wedge Y_V & \longrightarrow & Y_W \end{array}$$

This data is equivalent to a map $\bigvee_V S^{-V} \wedge X_V \to Y$ such that the two maps,

$$\bigvee_{V,W} S^{-W} \wedge I_G(V,W) \wedge X_V \rightrightarrows Y$$

coincide. In other words, to give a map $X \to Y$ is to give a map from the coequalizer:

$$\bigvee_{V,W} S^{-W} \wedge I_G(V,W) \wedge X_V \rightrightarrows \bigvee_V S^{-V} \wedge X_V$$

Thus X is this coequalizer. Moreover this description is functorial. (In fact it is a reflexive coequalizer). We will call this the *canonical presentation* (or *tautological presentation*?). To give a functor which preserves colimits, it is sufficient to define it on objects $\bigvee_V S^{-V} \wedge X_V$, and on these maps.

On homotopy day, we will learn how to write X (up to weak equivalence) as a filtered colimit of $S^{-V} \wedge X_V$. The same exercise allows you to write every Mackey functor as a coequalizer of representable ones. Another remark, for each V the functor $\tau^G \to \mathcal{S}^G$ from spaces to spectra sending $K \mapsto S^{-V} \wedge K$ is left adjoint to the functor sending $X \mapsto X_V$. Convention: $V = \{0\}$, then $S^{-V} = S^0$. $\Sigma^\infty : \tau^G \to \mathcal{S}^G$ is given by $\Sigma^\infty K = S^0 \wedge K$. It has a right adjoint Ω^∞.

Now we will try to make $\mathcal{S}^G$ into a symmetric monoidal category. The monoidal product will be $\wedge$. What properties do we want?

– $(-) \wedge (-)$ commutes with colimits in each varaible. (So there is an internal hom)

– $S^{-V} \wedge S^{-W} \cong S^{-(V \oplus W)}$

– $(X \wedge Y) \wedge K \cong X \wedge (Y \wedge K)$ where $X, Y \in \mathcal{S}^G$ and $K \in \tau^G$.

By the canonical presentation our desires force us to take $S^{-U} \wedge X$ to be the coequalizer

$$\bigvee_{V,W} S^{-U \oplus W} \wedge I_G(V,W) \wedge X_V \rightrightarrows \bigvee_V S^{-U \oplus V} \wedge X_V$$

Then $Y \wedge X$ is the coequalizer of

$$\bigvee_{V,W} S^{-W} \wedge I_G(V,W) \wedge Y_V \wedge X \rightrightarrows \bigvee_V S^{-V} \wedge Y_V \wedge X.$$

So the canonical presentation forces it to be given by this formula. There is a more coordinate free/global way to say this.

$$\begin{tikzcd}
I_G \times I_G \arrow[r, "{(X,Y)}"] \arrow[d, "\oplus"'] & \tau_G \times \tau_G \arrow[d, "\wedge"] \\
I_G & \tau_G
\end{tikzcd}$$

Then $X \wedge Y$ is the left Kan extension of $\wedge \circ (X,Y)$ along $\oplus$. This is sometimes called Day convolution. The unit for $\wedge$ is S^0. Also the functor $I^G \to \mathcal{S}^G$ sending V to S^{-V} is symmetric monoidal. (There is also a symmetric monoidal functor $I_G \to \mathcal{S}_G$, where the later has non-equivariant maps).

$\mathcal{S} = \mathcal{S}^G$ for $G = 1$ the trivial group.

Proposition 1.12.1. *$\mathcal{S}^G$ is equivalent to the category of objects in $\mathcal{S}$ equipped with a G-action, as a symmetric monoidal category.*

Example 1.12.2. $X \in \mathcal{S}$, then $X \wedge X$ with its $\mathbb{Z}/2$-action, defined a $\mathbb{Z}/2$-spectrum in $\mathcal{S}^{\mathbb{Z}/2}$. ◇

This is good for category theoretic purposes, but not so good for homotopy theoretic purposes. It can be very useful to have both points of view.

Proof. Let $I = I_G$ with $G = 1$ trivial. Then $I_G =$ objects in I equipped with a G-action.
Step 1: The category of objects in $\mathcal{S}$ with a G-action is the same as the category of functors from I to τ_G.

Step 2: $I \subset I_G$ as the full subcategory of those V with trivial G-action.

Step 3: By left kan extesnion we get a left adjoint $\mathcal{S}$ + G-action $\to \mathcal{S}^G$. These are equivalences of categories. □

For $X \in \mathcal{S}$ we get X_V and maps $I(V,W) \wedge X_V \to X_W$. If we have a G-action, it acts on on each X_V, but not on $I(V,W)$.

The key point: if U, V are two G-reps of the same dimension, then

$$O(V,W) \times O(V) \times O(U,V) \rightrightarrows O(V,W) \times O(U,V) \to O(U,W)$$

is a coequalizer diagram in I_G.

How to find S^{-V} in $\mathcal{S}$, where V has a non-trivial G-action. Answer: choose a U with trivial action, so that $\dim(U) = \dim(V)$. Then we have $S^{-U} \wedge O(U, V)$. In fact we have, S^{-V} is the coequalizer of

$$S^{-U} \wedge O(U,U)_+ \wedge O(V,U)_+ \rightrightarrows S^{-U} \wedge O(V,U)_+$$

1.13 DATE: Oct 5, 2011

[[★★★ Missed Oct 3, 2011 lecture. Covered: indexed tensor product]]

One more talk on category theoretic properties of the category $(\mathcal{S}^G, \wedge, S^0)$. Here all the contrsuctions come down to the question of isomorphism. On friday we will start exploring how these constructions interact with weak equivalences. There are some technical issues, some of which will be swept under the rug. But I don't know how big the broom is yet.

Distributive Laws

Suppose that $\mathcal{C}$ is a monoidal category with two monoidal structures $\otimes$ and $\oplus$. In all the cases we will consider, $\otimes$ will be a left adjoint and $\oplus$ will be the categoric coproduct, so that they distribute over each other. Let $p : I \to J$, and let $\Gamma =$ set of sections of $p : I \to J$.

$$\bigotimes_j \Big(\bigoplus_{p(i)=j} X_i \Big) = \bigoplus_\gamma \Big(\bigotimes_{j \in J} X_{\gamma(i)} \Big)$$

In the language of distribution laws, $q : J \to pt$, $ev : \Gamma \times J \to I$, $\pi : \Gamma \times J \to \Gamma$, $r : \Gamma \to pt$

$$\begin{tikzcd}
C^I \arrow[rr, "ev^*"] \arrow[d, "p_*^{\oplus}"'] & & C^{\Gamma \times J} \arrow[d, "\pi_*^{\otimes}"] \\
C^J \arrow[dr, "q_*^{\otimes}"'] & & C^{\Gamma} \arrow[dl, "r_*^{\oplus}"] \\
& C &
\end{tikzcd}$$

This is natural in the sets I, J, and pt, so it is automatically satisfied for covering categories as well:

Now we will generalize slightly. Suppose we have $p : I \to J$, and $q : J \to K$ are *covering categories* (they come from the Grothendieck construction with values in finite sets). Now we will form Γ to be the category of fiberwise sections of $I \to J$ over K. An object of Γ is a pair (k, s) consisting of $k \in K$ and $s : J_k \to I_k$, where I_k and J_k are the fibers over k. A section of $\Gamma \to K$ is a section of p.

$$\begin{tikzcd}
\Gamma \times_K J \arrow[rr, "ev"] \arrow[dr] & & I \arrow[dl] \\
& K &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
C^I \arrow[rr, "ev^*"] \arrow[d, "p_*^{\oplus}"'] & & C^{\Gamma\times J} \arrow[d, "\pi_*^{\otimes}"] \\
C^J \arrow[dr, "q_*^{\otimes}"'] & & C^{\Gamma} \arrow[dl, "r_*^{\oplus}"] \\
& C^K &
\end{tikzcd}$$

We will refer to this kind of argument as *working fiberwise*.

Algebras

A associative (commutative) algebra in $\mathcal{S}^G$ is an object $A \in \mathcal{S}^G$ equipped with a unit $S^0 \to A$ and a multiplication map $A \wedge A \to A$, which is unital, i.e.

$$S^0 \wedge A \to A \wedge A \to A$$
$$A \wedge S^0 \to A \wedge A \to A$$

are the canonical maps, and which is associative (and commutative). In any symmetric monoidal category $(C, \otimes, 1)$ you can consider $Comm(C)$ and $Ass(C)$ the category of commutative algebras in C and associative algebras in C. We are going to assume that C has all limits and colimits and $\otimes$ distributes over the categorical sum.

We can talk about these commuative and associative algera. We can ask about limits and colimits, adjoints, is there a free commutative/associative algebra? etc.

formal fact: $Comm(C)$ has a coproduct, and the coproduct of commutative algebras is given by $\wedge$ on the underlying objects. If I is a finite set, then the I-coproduct is given by the indexed monoidal product $\otimes_{i\in I}$. The coproduct $Comm(C)^I \to Comm(C)$ is left adjoint to the diagonal functor. More generally if $p : I \to J$ is a finite map of sets, then we have an adjunction

$$p_* : Comm(C)^I \leftrightarrows Comm(C)^J : p^*$$

This means that

$$\begin{tikzcd}
Comm(C)^I \arrow[r] \arrow[d, "p_*"'] & C^I \arrow[d, "p_*^{\otimes}"] \\
Comm(C)^J \arrow[r] & C^J
\end{tikzcd}$$

commutes up to natural isomorphism. Thus this also commutes up to natural isomorphism when $I \to J$ is a covering category.

So for example $i : H \hookrightarrow G$, then we have an adjunction:

$$i_* : \mathcal{S}^H \leftrightarrows \mathcal{S}^G : i^*$$

These is also a adjunction at the level of commutative algebras:

$$i_* : Comm(\mathcal{S}^H) \leftrightarrows Comm(\mathcal{S}^G) : i^*$$

Thus,

$$\begin{tikzcd} Comm(\mathcal{S}^H) \arrow[r] \arrow[d, "i_*"'] & \mathcal{S}^H \arrow[d, "norm_H^G"] \\ Comm(\mathcal{S}^G) \arrow[r] & \mathcal{S}^G \end{tikzcd}$$

[[★★★ The norm was discussed last time.]] $B_{pt}H \simeq B_{G/H}G \to B_{pt}G$. The we get

$$\mathcal{S}^H \simeq \mathcal{S}^{B_{pt}H} \overset{\simeq}{\leftarrow} \mathcal{S}^{B_{G/H}G} \overset{p_*^{\otimes}}{\to} \mathcal{S}^{B_{pt}G} \simeq \mathcal{S}^G$$

The composite is the norm $norm_H^G \simeq N_H^G$.

For $i \in G/H$, let $C_i \subset G$ denote the coset, and H_i for the stabilizer of i. Let $X \in \mathcal{S}^H$. Then $X_i = (C_i)_+ \wedge_H X$ (this is the coequalizer of $(C_i)_+ \wedge H_+ \wedge X \rightrightarrows (C_i)_+ \wedge X$) is an object of $\mathcal{S}^{H_i}$. Then we have,

$$\wedge_{i \in G/H} X_i = Norm_H^G(X)$$

where G acts on this. Because G is permuting the factors, you can't really break apart these factors. But by the langauge of covering categories we can still do these calculations and arguments.

1.14 Setting up the homotopy theory DATE: Oct 7th, 2011

We have an adjunction $\tau^G \leftrightarrows \mathcal{S}^G$ sending K to $S^0 \wedge K$, and sending $X \in \mathcal{S}^G$ to X_0.

- Equivaraintly stable
- Contains SW^G: If K and L are finite G CW-complexes, $ho\mathcal{S}^G(S^0 \wedge K, S^0 \wedge L) = \{K, L\}^G$.

Philosophy: The weak equivalences are fundamental. They determine the homotopy theory. There are many classes of fibrations and cofibrations which will be compatible with the weak equivalences. It is useful to be agnostic about which classes we are going to use until we really need them.

Stable Homotopy Groups

Choose a sequence $V_1 \subset V_2 \subset \cdots$ of G-representations such that every G-rep W embeds in V_i for sufficiently large i. $\underline{\pi}_k X$: finite G-sets $\to$ abelian groups. For S a finite G-set,

$$\underline{\pi}_k X(S) = \operatorname*{colim}_{n \to \infty} [S^k \wedge S^{V_n} \wedge S_+, X_{V_n}]^G$$

The inclusion $V_n \to V_{n+1}$ gives a point in $O(V_n, V_{n+1})$, and hence a sphere $S^{V_{n+1} - V_n} \subset I_G(V_n, V_{n+1})$. Then we get

$$S^{V_{n+1}-V_n} \wedge X_{V_n} \to I_G(V_n, V_{n+1}) \wedge X_{V_n} \to X_{V_{n+1}}$$

gives the transition maps. For $k \leq 0$, choose $\mathbb{R}^k \subset V_n$ and let

$$\underline{\pi}_k X(S) = \operatorname*{colim}_{n \to \infty} [S^{V_n - \mathbb{R}^n} \wedge S_+, X_{V_n}]^G$$

There is an alternative manifestly invariant version. J^G is the category where objects are orthogonal G-reps V. $J^G(V,W) = \pi_0(O(V,W)^G)$.

$$\underline{\pi}_k X(S) = \operatorname*{colim}_{V \in J^G} [S^k \wedge S^V \wedge S_+, X_V]^G$$

For sufficiently large W, $\pi_0 O(V,W)^G = pt$. This shows that these two are equivalent notions of stable homotopy groups.

The connectivity of $O(V,W)^G$

$V = \oplus_\alpha V_\alpha$ where $\alpha \in$ irreps of G, and V_α is the isotypical component. Similarly $W = \oplus_\alpha W_\alpha$ Then

$$O(V,W)^G = \prod_\alpha O(V_\alpha, W_\alpha)^G$$

So we can reduce to $O(U_\alpha^m, U_\alpha^n)^G$ with U_α irreducible. There is a fibration,

$$O(U_\alpha^{m-1}, U_\alpha^{n-1})^G \to O(U_\alpha^m, U_\alpha^n)^G \to O(U_\alpha, U_\alpha^n)^G$$

The bottom space is S^{n-1} or possibly a different (higher) dimensional sphere (depending on the type of real representation (real, complex, quaternionic)).

Weak equivalences

A (stable) weak equivalence a map $X \to Y$ in $\mathcal{S}^G$ is a *stable weak equivalence* if it induces an isomorphism $\underline{\pi}_* X \to \underline{\pi}_* Y$. This already determines $ho\mathcal{S}^G$. The morphisms are equivalences classes of diagrams,

$$X \xleftarrow{\sim} X_0 \to X_1 \xleftarrow{\sim} X_1' \to X_2 \xleftarrow{\sim} X_2' \to X_3 \cdots \xleftarrow{\sim} X_m' \to Y$$

where $\xleftarrow{\sim}$ denotes a stable weak equivalence.

Equivariantly stable

$(X \wedge S^V)_W = X_W \wedge S^V$ and $(\Omega^V X)_W = maps(S^V, X_W)$. $X \mapsto X \wedge S^V$ and $X \mapsto \Omega^V X$ are supposed to be inverse equivalences of functors on the homotopy category $ho\mathcal{S}^G$.

This will be a consequence of the following result.

Proposition 1.14.1. *The following are equivalent.*

1. *$X \to Y$ is a $\underline{\pi}_*$-isomorphism;*

2. *$\Omega^V X \to \Omega^V Y$ is a $\underline{\pi}_*$-isomorphism;*

3. *$S^V \wedge X \to S^V \wedge Y$ is a $\underline{\pi}_*$-isomorphism;*

This means we can calculate $S^V : ho\mathcal{S}^G \leftrightarrows h\mathcal{S}^G : \Omega^V$ by applying them to any object.

Proposition 1.14.2. *The maps $X \to \Omega^V(S^V \wedge X)$ and $S^V \wedge \Omega^V X \to X$ are $\underline{\pi}_*$-isomorphisms.*

Proof of Prop. 1.14.1 . (1) ⇒ (2). We know the following is a bijection.

$$\operatorname*{colim}_{W\to\infty}[S^k \wedge S^W \wedge S_+, X_W]^G \to \operatorname*{colim}_{W\to\infty}[S^k \wedge S^W \wedge S_+, Y_W]^G$$

We will introduce some temporary notation. If K is a finite pointed G-CW complex, then

$$\underline{\pi}_k(X)(K) \operatorname*{colim}_{W\to\infty}[S^k \wedge S^W \wedge K, X_W]^G.$$

Unfortunately $\underline{\pi}_k X(S) = \underline{\pi}_k X(S_+)$ (left is a finite G-set, right is a finite pointed G-CW complex). We will only use this notation in this lecture, and maybe the next. Please forgive me.

Note that $\underline{\pi}_* X \to \underline{\pi}_* Y$ isomorphism, then for all finite pointed G-CW complexes K we have $\underline{\pi}_* X(K) \to \underline{\pi}_* Y(K)$ is an isomorphism. Now $\underline{\pi}_k(\Omega^V X)(S_+) = (\underline{\pi}_k X)(S^V \wedge S_+)$, so (1) ⇒ (2).

We will come back to these in the next class. □

Proof that $X \to \Omega^V(S^V \wedge X)$ is a weak equivalence.

$$\begin{aligned}
\underline{\pi}_k(X)(S) &= \operatorname*{colim}_{W\to\infty}[S^k \wedge S^W \wedge S_+, X_W]^G \\
&\to \underline{\pi}_k(\Omega^V S^V \wedge X)(S) \operatorname*{colim}_{W\to\infty}[S^k \wedge S^W \wedge S_+, \Omega^V S^V X_W]^G \\
&= \operatorname*{colim}_{W\to\infty}[S^k \wedge S^{W\oplus V} \wedge S_+, S^V X_W]^G \\
&\to \operatorname*{colim}_{W\to\infty}[S^k \wedge S^{W\oplus V} \wedge S_+, X_{W\oplus V}]^G \\
&= \underline{\pi}_k(X)(S)
\end{aligned}$$

It is not hard to see that this is the identity. Similarly the other composite is the identity. □

$X \in \mathcal{S}^G$, then $(S^{-V} \wedge X)_W = O(V \oplus U, W)_+ \wedge_{O(U,U)_+} X_U$ where U is any G-rep with $\dim U + \dim V = \dim W$ (if $\dim W \geq \dim V$). If $\dim W < \dim V$, then a family of pointed spaces X_U paramertized by $O(V \oplus U, W)/O(U,U) = O(V,W)$. There is a family of X_{W-V} parametrized by $O(V,W)$.

1.15 DATE: Oct 12, 2011

We have this category τ^G of G-spaces and $\mathcal{S}^G$ of equivariant orthogonal spectra. And we have an adjunction between them. This induces an embedding of the Spanier-Whitehead category. $\underline{\pi_k} X = \operatorname{colim}_V \underline{\pi_{k+V}} X$, and

$$\underline{\pi_k} X(S) = \operatorname*{colim}_V [S^{k+V} \wedge S_+, X_V]^G$$

The weak equivalences are the isomorphisms on $\underline{\pi}_k$. Last time we had:

Proposition 1.15.1 (Prop 1.14.1)**.** *The following are equivalent.*

1. *$X \to Y$ is a $\underline{\pi}_*$-isomorphism;*
2. *$\Omega^V X \to \Omega^V Y$ is a $\underline{\pi}_*$-isomorphism;*

3. $S^V \wedge X \to S^V \wedge Y$ *is a* $\underline{\pi}_*$*-isomorphism;*

Moreover $X \to \Omega^V S^V \wedge X$ *is a weak equivalence, and* $S^V \Omega^V X \to X$ *is a weak equivalence.*

We had shown that (1) ⇒ (2) and (3) ⇒ (1), and the first of the second set of statements. The Key thing we will eventually show today is that (1) ⇒ (3).

Proposition 1.15.2. *For all* X *and all* V*, the map* $S^{-V} \wedge S^V \wedge X \to X$ *is a weak equivalence.*

Proof. We use the formula $(S^{-V} \wedge Y)_W = O(V \oplus U, W)_+ \wedge_{O(U,U)} Y_U$ where U is any G-rep with $\dim U + \dim V = \dim W$. We think of this later as a family of pointed G-spaces X_U parametrized by $O(V \oplus U, W)/O(U,U) \simeq O(V,W)$. We saw last time that $O(V,W)$ is equivariantly highly connected for $W >> 0$. So if we choose any point of $O(V,W)$, i.e. an equivariant embedding $V \to W$, the fiber over this point mapping to the total space

$$X_{W-V} \to O(V \oplus U, W)_+ \wedge_{O(U,U)} X_U$$

is an equivariant equivalnce in a given 'range' of dimensions.

Now applying this, for $W >> 0$,

$$\begin{aligned} \underline{\pi}_{k+W}(S^{-V} \wedge S^V \wedge X)_W &= \underline{\pi}_{k+W}(O(V \oplus U, W)_+ \wedge_{O(U,U)} S^V \wedge X)_W \\ &= \underline{\pi}_{k+W} S^V \wedge X_{W-V} \end{aligned}$$

so $\underline{\pi}_k(S^{-V} \wedge S^V \wedge X) = \mathrm{colim}_{W\to\infty}\, \underline{\pi}_{k+W}(S^{W-V} \wedge X_V) \cong \mathrm{colim}_{W\to\infty}\, \underline{\pi}_{k+W}(X_W) \cong \underline{\pi}_k(X)$. □

The point is that this is a formal consequence of the high connectivity of the spaces in this particular model of equivariant spectra.

Proposition 1.15.3. *If* K *is a pointed finite* G*-CW complex, then* $h\mathcal{S}^G(S^0 \wedge K, X) = \mathrm{colim}_W [S^W \wedge K, X_W]^G$*.*

[[★★★ Q:Why just finite? All G-CW are colimits of finite. Ans: You get a formula, but there is a $\lim^1$-term.]]

Corollary 1.15.4. $SW^G \subseteq h\mathcal{S}^G$*.*

Proof. Apply the formula to $X = S^0 \wedge L$ for a finite G-CW complex. □

Let $\mathcal{S}^G$ and let W be the collection of weak equivalences. Then $\mathcal{S}^G(X,Y)$ can be defined as the set of equivalence classes of zig-zags of maps. [[★★★ Drawing on board. Somewhat ambiguous.]] The leftward maps are weak equivalences. Modulo the equivalence relation generated by:

– replacing composable maps by their composition.

– $a \xrightarrow{f} b \xleftrightarrow{f} a$ and $a \xleftarrow{f} b \xrightarrow{f} a$

– identities are identities.

Claim 1.15.5. *Suppose we have* S *a finite* G*-set, and* X*,* Y *such that:*

$$\begin{tikzcd} S^{-(V\oplus W)} \wedge S^{V\oplus W+k} \wedge S_+ \arrow[r, dashed] \arrow[d, dashed] & X \arrow[d, "\simeq"] \\ S^{-V} \wedge S^{V+k} \wedge S_+ \arrow[r] & Y \end{tikzcd}$$

Then for $W >> 0$, the dashed arrows exist and the diagram commutes up to homotopy.

$\pi_0\mathcal{S}^G(S^{-V} \wedge S^{V+k} \wedge S_+, Y) = \pi_0\tau^G(S^{V+k} \wedge S_+, Y_V) = [S^{V+k} \wedge S_+, Y_V]^G$. Latter on in the system it comes from X.

Using this claim, we can shorten zig-zags of morphisms. This implies that for any finite G-set S, $h\mathcal{S}^G(S^0 \wedge S_+, X) = \operatorname{colim}_{W\to\infty} \pi_0\mathcal{S}^G(S^{-W} \wedge S^W \wedge S_+, X) = \underline{\pi}_0 X(S)$. Now work through an equivariant cell decomposition.

So we can see the Spanier-Whitehead category in $h\mathcal{S}^G$ even without invoking a model category structure.

Corollary 1.15.6. *$S^0 \wedge S_+$ is self-dual. And so $\underline{\pi}_0(X \wedge S_+)(T) = \underline{\pi}_0(X)(S \times T)$.*

This implies that $X \to Y$ a weak equivalence implies that $X \wedge S_+ \to Y \wedge S_+$ is a weak equivalence. Then, working through a CW-complex structure, we learn that $X \wedge K \to Y \wedge K$ is a weak equivalence for any pointed finite G-CW complex K. This uses: A sequence $A \to X \to X \cup CA$ gives a long exact sequence of $\underline{\pi}_*$ Mackey functors.

To prove this, it suffices to show exactness just at the X term. We can apply $S^{-V} \wedge S^V$ to these spaces (since it is a weak equivalence). Then we can look at the individual spaces and using the connectivity of Steifel manifolds, this follows from stability:

$$S^{W'} \wedge A_W \to S^{W'} \wedge X_W \to S^{W'} \wedge X_W \cup CA_W$$

is a fibration sequence in a range of dimensions.

Next time we will discuss some things that come up with symmetric products. Then we can probably black-box this machinery and start discussing the slice tower.

1.16 Families DATE: Oct 14, 2011

X a G-space such that X^H is either $\emptyset$ or contractible for each $H \subseteq G$. Let's look at all the set $\mathcal{F} = \{H \subseteq G \mid X^H \simeq pt\}$. Then if $H \in \mathcal{F}$,

- $H' \subset H \Rightarrow H' \in \mathcal{F}$
- $H \in \mathcal{F} \Rightarrow gHg^{-1} \in \mathcal{F}$.

A collection of subgroups of G satisfying these conditions is called a *family*. Given a family, can we build such an X? First we try to construct an S such that $S^H = \emptyset$ iff $H \notin \mathcal{F}$. This is easy, for example $S = \sqcup_{H\in\mathcal{F}} G/H$. To get X we can either take the infinite join with itself. We can also form the geometric realization of the pair groupoid on S. Either way we get a G-CW complex X satisfying this property.

We could also say that

- the space $Map^G(T, X)$ is empty or contractible. T is a finite G-set. The space is contractible if T has a point fixed by an element of $\mathcal{F}$, and empty otherwise.

The G=cells of an G-CW complex X satisfying this condition are of the form $T \times D^n$ where the stabilizer group of every point of T is an element of $\mathcal{F}$. This makes it easy to see that if X, X' are G-CW complexes satisfying the above property, then $Map^G(X, X')$ is contractible. Moreover $Map^G(K, X')$ is contractible if K is a G-CW-complex with such cells. So X is characterized up to a contractible space of choices. We write $X = E\mathcal{F}$.

It is common to take $\mathcal{P} = \{H \subsetneq G\}$ to be all proper subgroups. Let $\overline{\rho}$ be the reduced regular representation. Then $E\mathcal{P} = \operatorname{colim}_{n\to\infty} S(n\overline{n\rho})$.

This next maneuver was made famous by Gunnar Carlson. let $\tilde{E}\mathcal{F}$ be the mapping cone of $E\mathcal{F} \to pt$. It is the unreduced suspension of $E\mathcal{F}$, and is a pointed G-CW complex.

Proposition 1.16.1. *If X is a pointed Gspace and K a pointed G-CW complex, then (pointed maps) $[K, \tilde{E}\mathcal{P} \wedge X]^G \cong [K^G, X^G]$, and moreover $[K, \tilde{E}\mathcal{P} \wedge X]^H = 0$ for $H \subsetneq G$.*

Proof. Let T be a G-set where every element of T is stabilized by a proper subgroup if G.

$$\begin{array}{ccc} T \times S^{n-1} & \rightarrow & T \times D^n \\ \downarrow & & \downarrow \\ \mathrm{L} & \longrightarrow & L' \end{array}$$

Then $[L', \tilde{E}\mathcal{P} \wedge X]^G \cong [L, \tilde{\mathcal{P}} \wedge X]^G$ because they differ by maps like $[(T \times S^{n-1})_+, \tilde{E}\mathcal{P} \wedge X]^G \cong 0$. For example $[(G/H)_+, \tilde{E}\mathcal{P} \wedge X]^G = [pt, \tilde{E}\mathcal{P} \wedge X]^H = [pt_+, (\tilde{E}\mathcal{P})^H \wedge X^H] = 0$.

The result follows since K is obtained from K^G by attaching cells of the form $T \times D^n$ where all the isotropy groups of points in T are proper subgroups. $\square$

So smashing with $\tilde{E}\mathcal{P}$ takes you out of equivariant homotopy theory and into the ordinary homotopy theory of the fixed points.

The Stable Category

Now $X \in \mathcal{S}^G$, the fixed point functor $(-)^G$ is given by $X^G \in \mathcal{S}$ is given by restricting X to trivial representations and then taking fixed points. $\pi_0 X^G = (\underline{\pi}_0 X)(pt) = h\mathcal{S}^G(S^0, X)$. But these have bad homotopical properties. It is not invariant under weak equivalence. Instead X^G will be the fixed points of a fibrant replacement of X (the right derived functor). Then, for example $\pi_0(S^0)^G =$ the Burnside ring of G. This also has bad properties:

- $(X \wedge Y)^G \neq X^G \wedge Y^G$.
- $(S^0 \wedge K)^G \neq S^0 \wedge K^G$.

Santa did not get your letter.

The Geometric Fixed Point Functor

$\Phi^G(X) = (\tilde{E}\mathcal{P} \wedge X)^G$. This is much better behaved.

Proposition 1.16.2. *The properties hold:*

- $\Phi^G(X \wedge Y) \simeq \Phi^G(X) \wedge \Phi^G(Y)$.
- $\Phi^G(S^{-V}) \simeq S^{-V_0}$

This implies that $\Phi^G(\Sigma^\infty K) \simeq \Sigma^\infty K^G$.

The *canonical homotopy presentation.* Let $X \in \mathcal{S}^G$, and choose $V_0 \subset V_1 \subset \cdots$ eventually containing all G-reps.

$$S^0 \wedge X_{V_0} \leftarrow S^{-V_1} \wedge S^{V_1 - V_0} \wedge X_{V_0} \to S^{V_1} \wedge X_{V_1} \leftarrow S^{-V_2} \wedge S^{V_2 - V_1} \wedge X_{V_1} \to S^{-V_2} \wedge X_{V_2} \leftarrow \cdots$$

Then let $\widetilde{S^{-V_n} \wedge X_{V_n}}$ and the h-colimit of the first n-worth of this diagram. Then $\widetilde{S^{-V_n} \wedge X_{V_n}} \to \widetilde{S^{-V_{n+1}} \wedge X_{V_{n+1}}}$ and then,

$$\operatorname*{colim}_{n\to\infty} \widetilde{S^{-V_n} \wedge X_{V_n}} \to X$$

is a weak equivalence. This canonical homotopy presentation is very useful.

For example we have,

$$\begin{aligned} \underline{\pi}_k \tilde{E}\mathcal{F} \wedge S^{-V} \wedge X_V &= \operatorname*{colim}_{W\to\infty} [S^{k+W}, \tilde{E}\mathcal{P} \wedge I(V,W) \wedge X_W]^G \\ &= \operatorname*{colim}_{W\to\infty} [S^{k+W_0}, I(V_0,W_0) \wedge X_V^G] \\ &= \pi_k S^{-V_0} \wedge X_V^G. \end{aligned}$$

where V_0, W_0 are the fixed points. With more care this implies that $\Phi^G(S^{-V} \wedge X_V) = S^{-V_0} \wedge X_V^G$.

For any X, we have $\Phi^G X = \operatorname{colim}_{n\to\infty} S^{-(V_n)_0} \wedge X_{V_n}^G$. Moreover $\Phi^G(S^{-V} \wedge X_V \wedge S^{-W} \wedge X_W) = \Phi^G(S^{-V} \wedge X_V) \wedge \Phi^G(S^{-W} \wedge X_W)$. These properties prove the properties in the proposition.

Isotropy Separation Sequence

$$(E\mathcal{P})_+ \wedge X \to X \to \tilde{E}\mathcal{P} \wedge X$$

The last space can be understood of $\Phi^G X$. The first space is built from $(G/H)_+ \wedge X$ with H proper subgroup of G, and can be understood by induction.

1.17 Symmetric Powers DATE: Oct 17, 2011

There are a few more technical loose ends to collect together today. Then on Wednesday we will start the Slice Filtration. There are a lot of sneaky technical details that are going to be swept under the rug and you are just going to have to trust that I am not lying too much.

$X \in \mathcal{S}^G$, then look at $X^{\wedge n}/\Sigma_n = Sym^n X \in \mathcal{S}^G$. For example, the free commutative algebra generated by X is $Sym X = \vee_{n\geq 0} Sym^n X$.

Question: If $X \to Y$ is a weak equivalence, is $Sym^n X \to Sym^n Y$ a weak equivalence?

Example 1.17.1. $X = S^{-1} \wedge S^1 \to S^0 = Y$. Is the induced map of symmetric powers a weak equivalences? $(S^{-1} \wedge S^1)^{\wedge n} = S^{-n} \wedge S^n = S^{-\rho_n} \wedge S^{\rho_n}$, where Σ_n permutes the n-basis vectors of the permutation representation ρ_n of Σ_n (the defining representation). On the other hand, $(S^0)^{\wedge n} = S^0$ with trivial Σ_n-action.

$$(S^{-\rho_n} \wedge S^{\rho_n})/\Sigma_n \to (S^0)/\Sigma_n = S^0.$$

For $V >> 0$, $(S^{-\rho_n} \wedge S^{\rho_n})_V = I_G(\rho_n, V) \wedge S^{\rho_n}$, where $I_G(\rho_n, V)$ is a family of $S^{V-\rho_n}$ sphere parametrized by $O_G(\rho_n, V)$. This has an action by $\Sigma_n \times G$, and the Σ_n-action is free (away from the base point).

Already this is a little odd as one side has a free action and the other has a trivial action. The ultimate answer is that this is *not* a weak equivalence. $\diamond$

Let $\mathcal{F}$ be the family of subgroups of $\Sigma_n \times G$ consisting of all $H \subseteq \Sigma_n \times G$ such that $H \cap \Sigma_n = \{e\}$. Then we define $E_G\Sigma_n := E\mathcal{F}$. I.e. $E_G\Sigma_n$ is the total space of the universal G-equivariant Σ_n-bundle.

Proposition 1.17.2. *Suppose K is a pointed $\Sigma_n \times G$-space and the Σ_n-action is free away from the base point. Then the map $(E_G\Sigma_n)_+ \wedge K \to K$ is a $\Sigma_n \times G$-weak equivalence.*

Proof. $H \in \Sigma_n \times G$, then $((E_G\Sigma_n)_+ \wedge K)^H = (E_G\Sigma_n)_+^H \wedge K^H$ is a weak equivalence by construction of $E_G\Sigma_n$. $\square$

Proposition 1.17.3. *If $X \to Y$ in $\mathcal{S}^G$ is a weak equivalence, then $(E_G\Sigma_n)_+ \wedge_{\Sigma_n} X^{\wedge n} \to (E_G\Sigma_n)_+ \wedge_{\Sigma_n} Y^{\wedge n}$ is a weak equivalence.*

Proof. Suppose that S is a $\Sigma_n \times G$-set which has free Σ_n-action. Then $S_+ \wedge_{\Sigma_n} X^{\wedge n} \to S_+ \wedge_{\Sigma_n} Y^{\wedge n}$ is a weak equivalence. Therefore $W \wedge_{\Sigma_n} X^{\wedge n} \to W \wedge_{\Sigma_n} Y^{\wedge n}$ is a weak equivalence, where W is any $\Sigma_n \times G$-CW complex with free Σ_n-action. $\square$

Back to the example.

$$(E_G\Sigma_n)_+ \wedge_{\Sigma_n} (S^{-1} \wedge S^1)^{\wedge n} \xrightarrow{\sim} Sym^n(S^{-1} \wedge S^1)$$

since $((S^{-1} \wedge S^1)^{\wedge n})_V$ is Σ_n-free away from the base point.

$$\begin{array}{ccc} (E_G\Sigma_n)_+ \wedge_{\Sigma_n} (S^{-1} \wedge S^1)^{\wedge n} & \longrightarrow & (E_G\Sigma_n)_+ \wedge_{\Sigma_n} (S^0)^{\wedge n} \simeq S^0 \wedge (E_G\Sigma_n/\Sigma_n)_+ \\ \downarrow & & \\ Sym^n(S^{-1} \wedge S^1) & & \end{array}$$

So the map in the example is

$$Sym^n(S^{-1} \wedge S^1) \simeq S^0 \wedge (E_G\Sigma_n/\Sigma_n)_+ \to Sym^n(S^0) = S^0$$

is *not* a weak equivalence. This is were we need to introduce a model structure or some sort of similar structure to compute the correct homotopically meaningful symmetric product.

Positive Stable Model Structure

- The weak equivalences are the same. The isos on $\underline{\pi}_*$. We need fibrations and cofibrations.

-

$\mathcal{A}$ detect the fibrations: $\mathcal{A} = \{S^{-V} \wedge I_+^{n-1} \to S^V \wedge I_+^n \mid V^G \neq 0\} \cup \{-\}$ $X \to Y$ RLP w.r.t. this means $X_V \to Y_V$ is a Serre fibration, but only for those V such that $V^G \neq 0$. [[★★★ Nevermind. We voted to skip ahead and summarize this.]]

In the summary, in the positive model structure we build cofibrant objects by attaching cells of the form $S^{-V} \wedge S_+ \wedge S^{n-1} \to S^{-V} \wedge S_+ \wedge D_+^n$, where S is a finite G-set *where* $V^G \neq 0$. For example if K is a G-CW complex then $S^0 \wedge K$ is not cofibrant in the positive stable model structure. But $S^{-1} \wedge S^1 \wedge K$ is cofibrant.

Proposition 1.17.4. *If X is positive stable cofibrant, then $(E_G\Sigma_n)_+ \wedge_{\Sigma_n} X^{\wedge n} \to Sym^n X$ is a weak equivalence.*

This lets you set up the homotopy theory of commutative algebras.

Proposition 1.17.5. *$Sym : \mathcal{S}^G \leftrightarrows Comm(\mathcal{S}^G) : U$ creates a model structure on $Comm(\mathcal{S}^G)$, i.e. a map $R \to S$ in $Comm(\mathcal{S}^G)$ is a fibration or weak equivalence if and only if $UR \to US$ is such, in the positive stable model structure on $\mathcal{S}^G$.*

Remark 1.17.6. Because this homotopically invariant it will be equivalent to the notion E_∞-algebras. But it is more convenient to work with commutative algebras, both conceptually and because of the relation with the norm. ◇

$$\begin{tikzcd}
Comm(\mathcal{S}^H) \arrow[r, "U"] \arrow[d, "N_H^G"'] & \mathcal{S}^H \arrow[d, "N_H^G"] \\
Comm(\mathcal{S}^G) \arrow[r, "U"'] & \mathcal{S}^G
\end{tikzcd}$$

where N_H^G is the left adjoint of the restriction functor. Question: Is this correct homotopically? $UN_H^G R \cong N_H^G UR$ at the level of functors, but homotopically we need to compare $N_H^G\widetilde{UR}$ to $UN_H^G\tilde{R} \cong N_H^G UR \cong UN_H^G R$, where $\widetilde{UR} \to UR$ is a cofibrant replacement.

Proposition 1.17.7. *Suppose that $R \in Comm(\mathcal{S}^H)$ is cofibrant, and $\widetilde{UR} \to UR$ is a cofibrant replacement. Then*

$$N_H^G\widetilde{UR} \to N_H^G UR \cong UN_H^G R$$

is a weak equivalence.

Example: $K \in \mathcal{S}^G$ cofibrant. Then $SymK$ is a cofibrant comm algebra, but $USymK = S^0 \vee K \vee Sym^2 K \vee \cdots$ is *not* cofibrant.

Proposition 1.17.8. *$\Phi^G N_H^G X \simeq \Phi^H X$.*

Proof. Write X as a (filtered) homotopy colimit of $S^{-V_n} \wedge X_{V_n}$ (the canonical homotopy presentation). Then

$$\begin{aligned}
\Phi^G N_H^G X &\simeq \operatorname{hocolim} \Phi^G N_H^G (S^{-V_n} \wedge X_{V_n}) \\
&\simeq \operatorname{hocolim} \Phi^G S^{-Ind_H^G V_n} \wedge N_H^G X_{V_n} \\
&\simeq \operatorname{hocolim} S^{-(Ind_H^G V_n)^G} \wedge (N_H^G X_{V_n})^G \\
&\simeq \operatorname{hocolim} S^{-V_n^H} \wedge X_{V_n}^H \\
&\simeq \Phi^H X.
\end{aligned}$$

□

1.18 The Slice Filtration DATE: Oct 19,2011

Proposition 1.18.1. *Suppose that M is any Mackey functor, then there exists a spectrum $HM \in \mathcal{S}^G$ with the property that $\pi_k HM = M$ if $k = 0$ and 0 otherwise. HM is characterized up to weak equivalence by this property.*

Proof. Write M as the cokernel of $F_1 \to F_0 \to M$ where each F_i is a direct sum of representable Mackey functors. So $F_i(S) = \oplus_\alpha Burn_G(S, T^i_\alpha)$. Then $F_i(S) = h\mathcal{S}^G(S^0 \wedge S_+, \vee_\alpha S^0 \wedge T^i_\alpha)$. Notation, we'll write $[X,Y]^G$ for $h\mathcal{S}^G(X,Y)$. The map $F_1 \to F_0$ is realized by a map

$$\vee S^0 \wedge (T^1_\alpha)+ \to \vee S^0 \wedge (T^0_\alpha)_+ \to X$$

Let X be the cofiber. By the LES we have $\pi_0 X = M$, and $\pi_j X = 0$ for $j < 0$. Now we form a map onto π_1,

$$\vee S^1 \wedge S_+ \to X \to X_1$$

where the wedge product is over all finite G-sets S, and all maps $S^1 \wedge S \to X$. Then X_1 is the cofiber. $\pi_i X_1 = \pi_i X$ if $i \leq 0$, and $\pi_1 X_1 = 0$. We continue using $\vee S^2 \wedge S_+$, etc. to get X_k. Then we form $X_\infty = \operatorname{colim} X_k$. We have $HM = X_\infty$. This presentation shows that maps from HM to any other spectrum with this property is contractible, which prove the uniqueness up to weak equivalence. □

If K is a G-space, then $\underline{\pi}_n HM \wedge K$ is a homology theory. $\tilde{H}^n_G(K, M) = [K, S^n \wedge HM]$ is a cohomology theory.

Starting with any X, if we "cone off" all maps $S^m \wedge S_+ \to X$ for $m > n$, and repeat. Then we get

$$X \to Post^n(X)$$

which is an iso on π_k for $k \leq n$, and $\pi_k Post^n(X) = 0$ for $k > n$. The fiber of $Post^n X \to Post^{n-1} X$ is (a shift of) $H\underline{\pi}_n X$, an equivariant Eilenberg-Maclane spectrum. This isn't so useful if you want to calculate the homotopy groups.

We will introduce a variation on this known as the slice tower. The main ingredient is that we had a notion of 'spheres' or 'cells' and we had a notion of dimension. Then we can build a tower by coning off everything of dimension greater then n. This can be a crappy tower. It might not have an convergence properties. It is sort of formal. We will do a particular case.

G, and $K \subseteq G$, and let ρ_K be the regular representation of K. Let $\hat{S}(m, K) = G_+ \wedge_K S^{m\rho_K}$. This is like a bouquet of spheres wedged together. It is an indexed wedge of $S^{m\rho_k}$, indexed by G/K.

Definition 1.18.2. The set $\mathcal{A}$ of *slice cells* (for G) is the set of all $\hat{S}(m, K)$ and $\Sigma^{-1}\hat{S}(m, K)$. The dimension of $\hat{S}(m, K)$ is $m|K|$ and the dimension of $\Sigma^{-1}\hat{S}(m, K)$ has dimension $m|K|-1$. **Warning** $\Sigma^{-2}\hat{S}(m, K)$ will *not* behave like something of dimension $m|K| - 2$. ◇

[[★★★ Idea instead of the word 'cells'? It is not too late!]]

Definition 1.18.3. A G-spectrum Y is slice n-null if $\mathcal{S}_G(\hat{S}, Y)$ is equivaraintly contractible for all $\hat{S} \in \mathcal{A}$, $\dim \hat{S} \geq n$. We will denote this as $Y < n$ or $Y \leq n - 1$. Dually, X is slice n-positive if $\mathcal{S}_G(X, Y)$ is equivariantly contractible for all $Y \leq n$. We notate this as $X > n$. ◇

Notation: Instead of slice 0-positive, we will say *slice positive*. Instead of slice 0-null, we will say slice null. $\mathcal{S}^G_{>n}$ is the full subcategory of all $X \in \mathcal{S}^G$ with $X > n$, and similar for other notations $\mathcal{S}^G_{<n}$, $\mathcal{S}^G_{\leq n}$, etc.

Remark 1.18.4. $\mathcal{S}^G_{>n}$ is the smallest full subcategory of $\mathcal{S}^G$ containing the slice cells $\hat{S}$ of dimension $> n$ and satisfying the properties:

- closed under weak equivalence
- closed under arbitrary wedges of objects
- If $X \to Y \to Z$ is a cofiber sequence, and X, Y are in $\mathcal{S}^G_{>n}$, then $Z \in \mathcal{S}^G_{>n}$
- If $X \to Y \to Z$ is a cofiber sequence, and X, Z are in $\mathcal{S}^G_{>n}$, then $Y \in \mathcal{S}^G_{>n}$

i.e. it is closed under weak equivalence, homotopy colimits, and extensions. $\diamond$

The Slice Tower

$X \to P^n X \to P^{n-1}X$, Where $P^n X$ is obtained from X be inductively killing all maps from $\hat{S} \to X$, where $\hat{S}$ is a slice cell of dimension greater then n. We would like to say $X = X_0$, and then we look at all $\vee\hat{S} \to X_{k-1} \to X_k$, where you wedge over all $\hat{S} \to X_k$ with dimension $> n$. Then we would like to form the homotopy colimit X_∞ of the X_k. We would like $X_\infty = P^n X$. But we don't quite know $P^n X < n$. We know that $[\hat{S}, X_\infty]^G = 0$, but we want $[\hat{S} \wedge K, X_\infty]^G = 0$ for all G-CW complex K. (i.e. that $\mathcal{S}_G(\hat{S}, X_\infty)$).

Proposition 1.18.5. *The following are equivalent:*

1. $X < n$
2. $[\hat{S}, X]^G = 0$ *for all slice cells* $\hat{S}$ *of dimension* $> n$.

We will prove this next time. It will help tell us how these slice cells behave.
$P_n X \to X \to P^n X$
The fiber $P^n_n X \to P^n X \to P^{n-1}X$ is the n-slice of X. What these are is very interesting!

Proposition 1.18.6. $H \subseteq G$.

1. *If* $\hat{S}$ *is a slice cell for* G, *of dimension* m, *then* $\hat{S}$ *is a wedge of* H*-slice cells of dimension* m.
2. *If* $\hat{S}$ *is an* H*-slice cell of dimension* m, *then* $G_+ \wedge_H \hat{S}$ *is a* G*-slice cell of dimension* m.

Let $i^* : \mathcal{S}^G \to \mathcal{S}^H$ be induced by $i : H \hookrightarrow G$. Now we can look at $X \in \mathcal{S}^G$ and form the slice tower $\{P^n_G X\}$ and then apply i^*, or we can look at the H-slice tower of i^*X.

Proposition 1.18.7. $P^n_H i^* X \simeq i^* P^n_G X$.

Consequence: $H = 1$ implies that the underlying spectrum, $i^* P^n X$, is the Postnikov tower of X.

1.19 DATE: Oct 21, 2011

We have the problem that this part of the material was written up very carefully. The way you tell a story and the way you write a math paper are different. Now I can't get the story back. On the plus side notes don't have to be as good. You can look at the paper.

Slice cells: $G_+ \wedge_H S^{m\rho_H}$ of dimension $m|H|$, and $\Sigma^{-1} G_+ \wedge_H S^{m\rho_H}$ of dimension $m|H|-1$.

$H \subset G$. and there is the restriction functor i_H^*,

$$G_+ \wedge_H (-) : \mathcal{S}^H \leftrightarrows \mathcal{S}^G : i_H^*$$

Proposition 1.19.1. *If $X \in \mathcal{S}^G$ and $X > n$, then $i_H^* X > n$. If $Y \in \mathcal{S}^H$, and $y > n$, then $G_+ \wedge_H Y > n$. And similarly for '$>$'.*

Proof. $X < n$. We want to show that $\mathcal{S}_H(\hat{S}, i_H^* X)$ is equivariantly contractible. This is the same as showing that $\mathcal{S}^H(\hat{S} \wedge K, i_H^* X)$ is contractible for all K, but this is just $\mathcal{S}^G(G_+ \wedge_H (\hat{S} \wedge K), X)$, so we need to show that $G_+ \wedge_H (\hat{S} \wedge K)$ is $> n$. This is implied if we know $G_+ \wedge_H (\hat{S} \wedge S_+ \wedge S^t)$, where S is an H-set and $t \geq 0$. This later follows since $\hat{S} \wedge S_+$ is a wedge of slice cells of the same dimension.

This should be much much cleaner then I am making it. $\square$

Proposition 1.19.2. *(Wirtmuller-Isomorphism) There is a natural isomorphism $[X, G_+ \wedge_H Y]^G \cong [i_H^* X, Y]^H$. So these are ambidextrously adjoint. (We've seen this in the case Y is a G-space.)*

This then implies that if $Y \in \mathcal{S}^H$ and $Y < n$, and $\dim \hat{S} > n$, then $[\hat{S}, G_+ \wedge_H Y]^G = [i_H^* \hat{S}, Y]^H = 0$.

Proposition 1.19.3. *The following are equivalent:*

1. $X < n$
2. $[\hat{S} \wedge S^t, X]^G = 0$ *for all* $\dim \hat{S} > n$.
3. $[\hat{S}, X]^G = 0$ *for all* $\dim \hat{S} > n$.

Proof. The first are easy. We will prove (3) implies (2). We will induct on $|G|$. Assuming that $\forall \hat{S} > n$, $[\hat{S}, X]^G = 0$, we have that for all $H \subset G$ and $\hat{T} > n$ an H-slice cell, $[\hat{T}, X]^H = [G_+ \wedge_H \hat{T}, X]^G = 0$. If $H \subsetneq G$, then we know the space of maps $\mathcal{S}_H(\hat{S}, X)$ is equivariantly contractible.

This implies that all G-maps $[\hat{S} \wedge K, X]^G = 0$ if K is a G-space built from cells $S_+ \wedge D^n_+$ where S has no G-fixed points and $n \geq 0$. So we need to show that $[\hat{S} \wedge S^t, X]^G = 0$ for $t \geq 0$ and $\hat{S} > n$. Now $S^t \hookrightarrow S^{t\rho_G}$ as the fixed points. Let $T \to S^t \hookrightarrow S^{t\rho_G}$ be the fiber. $S^{t\rho_G}$ is obtained from S^t by attaching cells induced from proper subgroups. So $K = S^{t\rho_G}/S^t$ is a G-spectrum (space) built entirely from induced cells. For $t \geq 1$, then it is build from cells of dim > 0. Therefore $T = \Sigma^{-1}(S^{t\rho_G}/S^t)$ is also built from induced cells. Equivalently, T is the cofiber of $S^{t-1} \to S^{t\rho_G - 1}$. Hence $[\hat{S} \wedge T, X]^G = 0$ by the induction hypothesis. By the cofiber sequence we get.... if $\hat{S} = G_+ \wedge_K S^{m\rho_K}$

$$\hat{S} \wedge S^{t\rho_G} = ... = G_+ \wedge_K (S^{(m + t|G/K|)\rho_K}).$$

$\square$

Multiplicative properties of the Slice tower

From the last proof we learned that,

Proposition 1.19.4. *$S^{m\rho_G} \wedge (-) : \{slice\ cells\, \dim = k\} \to \{slice\ cells\, \dim = k + m|G|\}$ is a bijection.*

Corollary 1.19.5. *$S^{m\rho_G} \wedge P_{n+1}X \simeq P_{n+m|G|+1}(S^{m\rho_G} \wedge X)$ and $S^{m\rho_G} \wedge P^n X \simeq P^{n+m|G|}(S^{m\rho_G} \wedge X)$.*

1.20 DATE: Oct 24, 2011

$\mathcal{A}$ set of slice cells $\{G_+ \wedge_H S^{m\rho_H}, \sigma^{-1}G_+ \wedge_H S^{m\rho_H}\}$ where $H \subsetneq G$, and $m \in \mathbb{Z}$.

Suppose I have a collection $\mathcal{C}$ of G-spectra. Then *span*$\mathcal{C}$ is defined to be the smallest subcategory of $\mathcal{S}^G$ and containing $\mathcal{C}$ and closed under the following:

- weak equivalences;
- arbitrary wedges;
- If $X \to Y \to Z$ is a cofibration sequence, then X, Y in *Span*$\mathcal{C}$ implies Z in *Span*$\mathcal{C}$;
- If $X \to Y \to Z$ is a cofibration sequence, then X, Z in *Span*$\mathcal{C}$ implies Y in *Span*$\mathcal{C}$.

Then $\mathcal{S}^G_{\geq n} = Span\{\hat{S} \in \mathcal{A} \mid \dim \hat{S} \geq n\}$. $Y \in \mathcal{S}^G_{<n} \Leftrightarrow \mathcal{S}_G(\hat{S}, Y) \simeq pt$ for each $\dim \hat{S} \geq n$.

$$P_{n+1}X \to X \to P^n X$$

The first is the universal map from something $\geq n+1$, and the latter is the universal map into something $\leq n$.

Exercise 1.20.1. Suppose that there is a cofibration sequence $\tilde{P}_{n+1}X \to X \to \tilde{P}^n X$ and $\tilde{P}^n X \leq n$, and $\tilde{P}_{n+1}X \geq n+1$. Then the sequence $\tilde{P}_{n+1}X \to X \to \tilde{P}^n X$ is equivalent to the slice sequence.

$P^n_n X =$ the fiber of $P^n X \to P^{n-1}X$ and also the cofiber of $P_{n+1}X \to P_n X$. It is *the n-slice of X*.

Slice connectivity and ordinary connectivity

If X is a filtered spectrum then GrX is the wedge of the associated graded.

Lemma 1.20.2. *Suppose that $\hat{S}$ is a slice cell of $\dim \hat{S} = n \geq 0$. Then $Gr\hat{S}$ is a wedge of things in $\{S_+ \wedge S^k\}$ where S is a finite G-set and $\lfloor n/|G| \rfloor \leq k \leq n$. If $\dim \hat{S} = n < 0$, then $Gr\hat{S}$ is a wedge of things in $\{S_+ \wedge S^k\}$ where S is a finite G-set and $n \leq k \leq \lfloor n/|G| \rfloor$.*

Proof. For $n \geq 0$ $\hat{S} = G_+ \wedge_H S^{m\rho_H}$ or $G_+ \wedge_H \sigma^{-1} S^{m\rho_H}$, the result follows from the fact that $S^{m\rho_H}$ has an H-CW decomposition: $S^m \cup$ higher cells up to $\dim H$. Case $n < 0$ follows from Spanier-Whitehead duality. □

Corollary 1.20.3. *If $X \geq n \geq 0$, then $X \in Span\{S_+ \wedge S^k \mid \lfloor n/|G| \rfloor \leq k \leq n\}$.*

Corollary 1.20.4. *$X \geq n$, and $n \geq 0$, then X is $(\lfloor n/|G| \rfloor - 1)$-connected. If $X \geq n$ and $n < 0$, then X is $(n-1)$-connected.*

Example: Fix k. For $n >> 0$, $\underline{\pi}_k X \to \underline{\pi}_k P^n X$.

Analyze $\underline{\pi}_k P^n X$ for $k >> n$. Q: where in the slice filtration is S^t of $S_+ \wedge S^t$, where S is a finite G-set?

Case I: S^t, $t \geq 0$.

Claim 1.20.5. *$S^t \geq t$.*

Proof. Induct on $|G|$. Can assume if $K \in Span\{S_+ \wedge S^\ell\}$ ranging over all $\ell \geq t$ and S with no fixed points, then $K \geq t$. We have:

$$T \to S^t \to S^{t\rho_H}$$

where T is such a cell. Thus $T \geq t$, and since $S^{t\rho_G} \geq t|G|$, we get $S^t \geq t$. □

Case II: Suppose $m \leq -1$, and $k \geq m$, then $S_+ \wedge S^k \geq (m+1)|G| - 1$. ($k \geq m$ follows from $k = m$). So we may assume that $k < 0$. Trick: write $S_+ \wedge S^k = S^{-1} \wedge S_+ \wedge S^{(k+1)\rho_G} \wedge S^{-(k+1)\overline{\rho}_G}$. Where $1 \oplus \overline{\rho}_G = \rho_G$. Now $S^{-1} \wedge S_+ \wedge S^{(k+1)\rho_G} \geq (k+1)|G| - 1$, and $S^{-(k+1)\overline{\rho}_G} \geq 0$, and so the bound follows.

Corollary 1.20.6. *For fixed k, there exists and n, such that $X < n$ implies that $\underline{\pi}_t X = 0$, for $t \geq k$.*

Corollary 1.20.7. $\lim P_{n+1} X \simeq pt$, *colim* $P^n X \simeq pt$, *and* $X \simeq \lim P^n X$.

The Slice Spectral Sequence

(this can be thought of as a Mackey functor values sepctral sequence, or a spectral sequence valued Mackey functor.)

$$E_2^{s,t} = \underline{\pi}_{t-s} P_t^t X \Rightarrow \pi_{t-s} X.$$

where the differential $d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$.

[[★★★ Illustrative image.]]

It is usually displayed with x-axis $= t - s$. The groups of a single slice occur along slope (-1) lines.

[[★★★ Illustrative image.]] Line through the origin with slope $|G| - 1$. Shaded regions show possible ranges of non-zero homotopy groups for slices.

Remark 1.20.8. $P_{-1}^{-1} X = (-1)$-Postnikov section. $P_0^0 = HM$ where M is a Mackey functor all of whose restriction maps are monomorphisms. ◇

Next time we will prove this.

1.21 DATE: Oct 26, 2011

No Class on Friday. Today is supposed to be kinda fun. We are going to try and identify some simple slices.

Proposition 1.21.1. *$X \geq 0$ iff X is (-1)-connected.*

Proof. S_+ is a wedge of slice cells of dimension zero, when S is a finite G-set. So X (-1)-connected $\Leftrightarrow X \in Span\{S_+\}$. □

Proposition 1.21.2. *$X \geq -1$ iff X is (-2)-connected.*

Proof. Same: $\Sigma^{-1}S_+$ is a wedge of -1-cells. □

$$P_0 X \to P_{-1} X \to P_{-1}^{-1} X$$

So the (-1)-slice of X is the (-1)-Postnikov section. So...

Corollary 1.21.3. *X is a (-1)-slice iff $X \simeq \Sigma^{-1} HM$ where M is any Mackey functor.*

To analyze $X > 0$, we need a

Lemma 1.21.4. *$X \in \mathcal{S}^G$ and for all $H \subsetneq G$, $i_H^* X > d$ in $\mathcal{S}^H$. Then $E\mathcal{P}_+ \wedge X > d$, where $\mathcal{P}$ is the family of proper subgroups and $E\mathcal{P}^H$ is pt if $H \subsetneq G$ and is empty for $H = G$.*

Recall,

$$E\mathcal{P}_+ \to S^0 \to \tilde{E}\mathcal{P}$$

Lemma 1.21.5. *$\tilde{E}\mathcal{P} \geq |G| - 1$.*

Proof. $S^{\rho_G - 1} \geq |G| - 1$. So $\tilde{E}\mathcal{P}_+ \wedge S^{\rho_G - 1} \geq |G| - 1$. But

$$\tilde{E}\mathcal{P} \to \tilde{E}\mathcal{P} \wedge S^{\rho_G - 1}$$

is an equivalence of G-spaces. We deduce this by looking at the fixed point spaces. Recall: $(\tilde{E}\mathcal{P})^H = pt$ if $H \subsetneq G$, and S^0 if $H = G$. The version for spectra follows from the space level equivalence. □

Proposition 1.21.6. *$X > 0$ iff X is (-1)-connected and $\pi_0^u X = 0$, where π_0^u means π_0 of the underlying non-equivariant spectrum.*

Proof. $X > 0 \Rightarrow X$ is (-1)-connected and $\pi_0^u X = 0$ is easy. For the converse,

$$E\mathcal{P}_+ \wedge X \to X \to \tilde{E}\mathcal{P}$$

X is (-1)-connected ($\Rightarrow X \geq 0$). The later is $\geq |G| - 1 \geq 1$ (the trivial group case is trivial). By induction on $|G|$ we can assume $i_H^* X > 0$ for all $H \subsetneq G$. Hence $EP_+ \wedge X > 0$, and therefore $X > 0$. □

Corollary 1.21.7. *If $S \to S'$ is a surjective map of finite G-sets, then $S_+ \to S'_+ \to S'_+ \cup CS_+$, and $S'_+ \cup CS_+ > 0$.*

Proof. $S'_+ \cup CS_+ \geq 0$ and $\pi_0^u S'_+ \cup CS_+ = 0$. □

From yesterday's lecture, if X is a zero slice, then $X = HM$ for *some* M. In fact...

Corollary 1.21.8. *HM is a 0-slice iff for all surjective maps $S \to S'$, the map $M(S') \to M(S)$ is a monomorphism.*

Proof. Look at the slice tower. $P_1 HM \to HM \to P^0 HM$. Let's look at the long exact sequence of homotopy groups.

$$0 \to \underline{\pi}_0 P_1 HM \to M \to \pi_0 P^0 HM \to 0$$

Let's notate them as:

$$0 \to M' \to M \to M'' \to 0.$$

We need to show that $M' = 0$. Now $M(S') \to M(S)$ a mono for all $S \to S'$ is equivalent to $M(S') \hookrightarrow M(G \times S')$, as there is:

$$\begin{tikzcd} & S \arrow[d] \\ G \times S' \arrow[ur, dashed] \arrow[r] & S' \end{tikzcd}.$$

Now a digram chase proves the result (look at $S \to G \times S$). We use that $M'(G \times S) = \pi_0 P_1 HM(G \times S) = \pi_0^u P_1 HM(S) = 0$ since $P_1 HM > 0$. $\square$

Theorem 1.21.9. *$P_0^0 S^0 = H\underline{\mathbb{Z}}$. ($\underline{\mathbb{Z}}$ is the constant Mackey functor).*

Proof. Let A be the Burnside Mackey functor $\underline{\pi}_0 S^0$. Then the fiber of $S^0 \to HA$ is > 0 (in fact simply connected). So this map is an equivalence:

$$P_0^0 S^0 = P^0 S^0 \to P^0 HA = P_0^0 HA.$$

So it suffices to show that $P_0^0 HA = H\underline{\mathbb{Z}}$.

$$I \hookrightarrow A \to \underline{\mathbb{Z}}$$

applied to the G-set G gives $0 \to \mathbb{Z} \to \mathbb{Z}$. This means that for $HI \to HA \to H\mathbb{Z}$, we have $\pi_0^u HI = 0$. But also $HI \geq 0$. Hence $HI > 0$. Hence $P^0 HA \to P^0 H\underline{\mathbb{Z}} \simeq H\underline{\mathbb{Z}}$ is an equivalence. $\square$

Remark 1.21.10. K a G-space, then $Sym^\infty K = \Omega H\underline{\mathbb{Z}} \wedge K$. (equivariant Dold-Kan). $\diamond$

Corollary 1.21.11. *If W is a wedge of slice cells of dimension d, then $W \wedge H\underline{\mathbb{Z}}$ is a d-slice.*

Proof. $W =$ wedges of $G_+ \wedge_H S^{m\rho_H}$ or $\Sigma^{-1} G_+ \wedge_H S^{m\rho_H}$. So it suffices to check that $S^{m\rho_H} \wedge H\underline{\mathbb{Z}}$ is a d-slice and $S^{m\rho_H} \wedge \Sigma^{-1} H\underline{\mathbb{Z}}$.

The result follows from the fact that $H\underline{\mathbb{Z}}$ is a zero slice and $\Sigma^{-1} H\underline{\mathbb{Z}}$ is a (-1)-slice. $\square$

Corollary 1.21.12. $P_{m|H|}^{m|H|} G_+ \wedge_H S^{m\rho_H} = H\underline{\mathbb{Z}} \wedge G_+ \wedge_H S^{m\rho_H}$.

Definition 1.21.13. A d-slice is *cellular* if it is of the form $H\underline{\mathbb{Z}} \wedge W$, where W is a wedge of slice cells of dimension d. We will say it is *pure* if W is a wedge of slice cells of the form $G_+ \wedge_H S^{m\rho_H}$.

A spectrum X is *pure* (or has pure slices) if $P_d^d X$ is pure for all d. $\diamond$

The main theorem that proves the Kervaire theorem is that a certain spectrum is pure.

1.22 DATE: Oct 31, 2011

Recall, a *cellular d-slice* is $H\mathbb{Z} \wedge \hat{W}$, where $\hat{W}$ is a wedge of slice cells of dimension d. Here $H\mathbb{Z}$ corresponds to the constant Mackey functor. Then X is *pure* if the slices are cellular with wedges of the form $G_+ \wedge_H S^{m\rho_H}$ (not desuspensions of these).

Example 1.22.1. $G = \mathbb{Z}/2$. Then the cellular slices are $H\mathbb{Z} \wedge S^{m\rho_2}$, $H\mathbb{Z} \wedge (\mathbb{Z}/2)_+ \wedge S^m$, and $H\mathbb{Z} \wedge S^{(m\rho_2)-1}$. Pure only uses the first two cases. For example: real K-theory, $K\mathbb{R}$, the odd slices are contractible, and the even slices are $H\mathbb{Z} \wedge S^{m\rho_2}$. It is pure. ◇

Suppose that X is pure, and $G \neq 1$. Want to study $\underline{\pi}_i X$ for $-4 < i < 0$. So we need to study $\underline{pi}_i H\mathbb{Z} \wedge (G)_+ \wedge_H S^{m\rho_H}$. Now

$$\underline{pi}_i^G H\mathbb{Z} \wedge (G)_+ \wedge_H S^{m\rho_H} \cong \pi^H \wedge S^{m\rho_H}$$

so we might as well study $\underline{pi}_i^G H\mathbb{Z} \wedge_H S^{m\rho_G}$. This is obviously zero for $m \geq 0$.

$$[S^{-i}, H\mathbb{Z} \wedge S^{-m\rho_G}]^G = [S^{m\rho_G}, S^i \wedge H\mathbb{Z}]^G = \tilde{H}^i_G(S^{m\rho_G}; \underline{\mathbb{Z}}) = \tilde{H}^i_G(S^{m\rho_G}/G; \underline{\mathbb{Z}})$$

for $0 < i < 4$. Now $S^{m\rho_G} = \Sigma^m S^{m(\rho_G - 1)}$. So

$$S(m(\rho_G - 1))/G \to pt \to S^{m(\rho_G-1)}/G$$

The first space is connected (if $G \neq \mathbb{Z}/2$), so the last space is simply connected. So

$$\tilde{H}^i_G(S^{m\rho_G}; \underline{\mathbb{Z}}) \cong \tilde{H}^{i-m}_G(S^{m(\rho_G-1)}; \underline{\mathbb{Z}}) \cong \tilde{H}^{i-m}(S^{m(\rho_G-1)}/G; \underline{\mathbb{Z}})$$

for $0 < i < 4$, we have $-m < i - m < 4 - m$. For $m \geq 2$, H^0, and H^1 are both zero by simple connectivity. The higher groups are also zero. This also applies when $m = 1$. So the only remaining case is when $m = 1$, for H^2. By simple connectivity, $H^2(S^{\rho_G - 1}; \mathbb{Z})$ injects into $H^2(S^{\rho_G-1}; \mathbb{Q})$. If G is finite, then $H^2(S^{\rho_G-1}/G; \mathbb{Q}) = H^2(S^{\rho_G-1}/G; \mathbb{Q}) = 0$ unless $|G| = 3$. If the order of G is three, then $H^2(S^{\rho_G-1}/G; \mathbb{Z}) = \mathbb{Z}$.

Summarizing, if $G \neq 1, C_3$, (and $m \neq 1$), then $\underline{\pi}_i^G H\mathbb{Z} \wedge S^{m\rho_G}$. This leads to the Gap Theorem.

Proposition 1.22.2. *Suppose that $X \to Y$ is a map of d-slices, and X is cellular. Then $d \not\equiv -1$ mod p where p divides $|G|$. Then if $\pi^u_d X \to \pi^u_d Y$ is an isomorphism then f is a weak equivalence. (the underling map of spectra)*

Example 1.22.3. $K\mathbb{R}$, $\pi^u_{2n} K\mathbb{R} \cong \mathbb{Z}$ with $(-1)^n$-action of $\mathbb{Z}/2$. There is an equivariant map $S^{n\rho_2} \to K\mathbb{R}$, whose underlying non-equivariant map is the generator. If $S^{n\rho_2} \geq 2n$, then $S^{n\rho_2} to P_{2nK\mathbb{R}}$ gives $H\mathbb{Z} \wedge S^{n\rho_2} \to P^{2n}_{2n} K\mathbb{R}$. By the proposition this is an equivalence. ◇

Example 1.22.4. Suppose that R is a C_4 spectrum and suppose that $\pi^u_4 R = \mathbb{Z}\langle b\rangle \oplus \mathbb{Z}\langle c\rangle \oplus \mathbb{Z}\langle a\rangle$ where the last $\mathbb{Z}$ has the sign representation, and the first two $\mathbb{Z}$'s form the 2-dimensional permutation representation. Suppose we guess $P^4_4 R$ is cellular. The 4D slice cells are $(C_4)_+ \wedge S^4$, $(C_4)_+ \wedge_{C_2} S^{2\rho_2}$, and S^{ρ_4}. By the proposition, if we can find maps $S^{\rho_4} \to R$ refining c to a C_4-equivariant map, and $S^{2\rho_2} \to R$ refining a to a C_2-equivariant map, then

$$P^4_4 R \simeq H\mathbb{Z} \wedge (S^{\rho_4} \vee (C_4)_+ \wedge_{C_2} S^{2\rho_2}).$$

◇

Proof. Define C as the mapping cone $X \to Y \to C$. Then $C \geq d$. We need to show that $[\hat{S}, C]^G = 0$ when $\hat{S}$ is a slice cell and $\dim \hat{S} \geq d$. By induction on the order of the group, we can assume that G is not trivial, and $\hat{S} \simeq S^{m\rho_G}$ or $S^{mrho_G - 1}$.

Case 1: $d \equiv 0 \bmod |G|$. By smashing with $S^{-\frac{|G|}{d}\rho_G}$, we can assume that $d = 0$. So in this case, X and Y are zero slices. $\underline{\pi}_0 X = M$, and $\underline{\pi}_0 Y = M'$, and $X = H\mathbb{Z} \wedge T_+$ where T is a discrete G-set. This implies that $M(S) = M(S \times G)^G$.

$$\begin{tikzcd} M(S) \arrow[r] \arrow[d, "\simeq"'] & M'(S \times G)^G \arrow[d, hook] \\ M(S \times G)^G \arrow[r, "\simeq"'] & M'(S) \end{tikzcd}$$

implies the top is an equivalence.

Case 2: $d \not| |G|$, then $\pi^G_{m\rho_G} C = 0$ for $m|G| > d$, and $\pi^G_{m\rho_G - 1} C = 0$ for $m|G| > d$. $\pi^G_V X - [S^V X]^G$. Low $m|G| - 1 > d$. and we have

$$0 = \pi^G_{m\rho_G} Y \to \pi^G_{m\rho_G} C \to \pi^G_{m\rho_G - 1} X = 0$$

so the middle term is zero. [...] □

claim If X is a cellular d-slice and $m|G| - 1 > d$ then $\pi^G_{m\rho_G - 2} \to \pi^u_{m|G|-2} X$ is a monomorphism.

1.23 Complex cobordism and formal groups DATE: Nov 2nd, 2011

We've discussed the slice tower and certain techniques for figuring out certain slices. We figured out the 0-slice of S^0, and we have the notion of a pure spectrum. Now we are going to start introducing some examples of spectra. The ones we are really going to work with.

Let MU be the Thom spectrum over BU of the universal virtual bundle V^0. $BU = \cup_n BU_n$, then $MU(n) = Thom(BU_n, V^n_{univ})$. We get $\Sigma^2 MU(n-1) \to MU(n)$. And $\operatorname{colim} \Sigma^{-2n} MU(n) = MU$. This has some interesting universal properties. MU is a commutative ring (BU is an infinite loop space).

Definition 1.23.1. Let E be a ring spectrum ($E \wedge E \to E$ is unital and homotopy associative). A complex orientation of E is an element $x \in \tilde{E}^2(\mathbb{CP}^\infty)$ such that x restricts to $1 \in E^0(pt)$, under

$$E^2(\mathbb{CP}^\infty) \to \tilde{E}^2(\mathbb{CP}^2) = \tilde{E}^2(S^2) \cong E^0(pt).$$

◇

Proposition 1.23.2. *If $x \in \tilde{E}^2(\mathbb{C}]P^\infty)$ is a complex orientation, then $E^*(\mathbb{CP}^\infty) = \pi_* E[[x]]$. More generally,*

$$E^*(\underbrace{\mathbb{CP}^\infty \times \cdots \times \mathbb{CP}^\infty}_{n \text{ times}}) = \pi_* E[[x_1, \ldots, x_n]]$$

where x_i comes from the image of x under the projection to i^{th}-factor.

Proof. By induction on n we show, $E^*(\mathbb{CP}^n) = \pi_*E[x]/x^{n+1}$. There is a morphism $\pi_*E[x]/x^{n+1} \to E^*(\mathbb{CP}^n)$, and it is an isomorphism for $n = 1$. Then we study the sequence

$$\to E^*(\mathbb{CP}^n, \mathbb{CP}^{n-1}) \to E^*(\mathbb{CP}^n) \to E^*(\mathbb{CP}^{n-1}) \to$$

and use $\mathbb{CP}^n/\mathbb{CP}^{n-1} \simeq S^{2n}$, for the inductive step. We use a factorization of Δ, $\mathbb{CP}^n \to S^{2n} \to \wedge^n \mathbb{CP}^n$ where the second map is degree 1. [[★★★ didn't quite follow.]] □

The map $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ gives us,

$$f(x, y) \in \pi_*E[[x, y]].$$

This defines a formal group law over π_*E. (Unital, commutative, and associative). Some of our applications are calculational so I will stick to formula based descriptions. But there are more conceptual and elegant ways to think about this.

There is a universal formal group law $F(x, y) = x + y + \sum_{i,j>1} a_{ij}x^iy^j$. $a_{ij} = a_{ji}$ and associativity gives a series of complicated relations. The *Lazard ring* is $\mathbb{Z}[a_{ij}]/$ these relations. Then

$$Ring(L, R) \leftrightarrow \text{Formal Group Laws over } R.$$

In fact there is a category of formal group laws.

Back in topology... MU itself is a complex oriented spectrum. $MU(1) = Th(BU(1), V^1_{univ}) = Th(\mathbb{CP}^\infty, L)$ which is the mapping cone of $pt \simeq S^\infty \simeq S(L) \to \mathbb{CP}^\infty \to Thom(\mathbb{CP}^\infty, L)$.

$$\Sigma^\infty \mathbb{CP}^\infty = MU(1) \to \Sigma^2 MU$$

gives $x \in \widetilde{MU}^2(\mathbb{CP}^\infty)$ and is a complex orientation.

Suppose that E is complex oriented. Then $E_*(\mathbb{CP}^\infty)$ is a free π_*E-module on $b_0, b_1, \ldots$ where each $b_i \in E_{2i}\mathbb{CP}^\infty$, where $\langle b_i, x^j \rangle = \delta_{ij}$. Now $L : \mathbb{CP}^\infty \to BU$, and we have a Schubert celll decomposition of BU (coming from Grassmanians) implies that $E_*(BU) = \pi_*E[b_0, b_1, \ldots]/(b_0 = 1)$. So

$$E_*BU = Sym_{E_*}(\tilde{E}_*\mathbb{CP}^\infty).$$

Similarly $E^*(BU) = \mathrm{Hom}_{\pi_*E}(E_*BU, \pi_*E)$.

Let $\beta_n \in E_{2n}\Sigma^2\mathbb{CP}^\infty$ correspond to b_{n-1} under the suspension isomorphism. Similar arguments show that $E_*MU = \pi_*E[\beta_1, \beta_2, \ldots]$, and $E^*MU = \mathrm{Hom}_{\pi_*E}[E_*MU, \pi_*E]$. So $E^*MU = [MU, E]$. Homotopy multiplicative maps are in bijection with ring maps $E_*MU \to E_*$.

Theorem 1.23.3 (Quillen)**.** *The map $L \to \pi_*MU$ sending a_{ij} to coefficients of the formal group law in $\pi_{2(i+j)}MU$ is an isomorphism.*

Theorem 1.23.4 (Lazard)**.** $L = \mathbb{Z}[x_1, x_2, \ldots]$ *with* $|x_i| = 2i$ *when* $|a_{ij}| = 2(i + j)$.

Real Bordism

Landweber, Araki, then later Hu-Kriz. Let's work in C_2-spectra. Consider $\mathbb{CP}^\infty$. C_2 acts on this by complex conjugation (the fixed point space is $\mathbb{RP}^\infty$). If $BU = \operatorname{colim} BU(n)$, then C_2 acts on this by complex conjugation. The fixed point space is BO. Araki defined the notion of a *Real oriented spectrum.* This is a C_2-spectrum with an element $\overline{x} \in \tilde{E}^{\rho_2}\mathbb{CP}^\infty$, where ρ_2 is the 2-dimensional sign representation. Then it restricts to $1 \in \tilde{E}^{\rho_2}(\mathbb{CP}^1) = \tilde{E}^{\rho_2}(S^{\rho_2}) = E^0(pt)$.

All the Schubert cells lift to C_2-equivariant cells with $S^{n\rho_2}$ replacing S^{2n}. So all of those previous computations go through. $E^\star(X) = \oplus_V E^V(X)$ is the $RO(G)$-graded cohomology, where $E^V(X) = [X, S^V \wedge E]^G$. Then,

$$E^\star(\mathbb{CP}^\infty) = (\pi_\star E)[[\overline{x}]].$$

The same argument works for these and all the same computations go through.

$$E^\star(\underbrace{\mathbb{CP}^\infty \times \cdots \times \mathbb{CP}^\infty}_{\text{n times}}) = \pi_\star E[[\overline{x}_1, \ldots, \overline{x}_n]]$$

Then we define Real bordism as $MU_\mathbb{R} = \operatorname{colim} S^{-n\rho_2} MU_\mathbb{R}(n)$, where $MU_\mathbb{R}(n)$ is the Thom space of the universal bundle over $BU(n)$, which the complex conjugation action. We get a map,

$$L \to \pi_\star MU_\mathbb{R}$$

sending a_{ij} to $a_{ij} \in \pi_{(i+j)\rho_2} MU_\mathbb{R}$.

Now $L = \mathbb{Z}[x_1, x_2, \ldots]$, and the image in $\pi_\star MU_\mathbb{R}$ is $\overline{x}_i \in \pi_{\rho_2} MU_\mathbb{R}$. For any monomial x_I, we get $\overline{x}_I \in \pi_{|I|\rho_2} MU_\mathbb{R}$. Using these we can construct,

$$\bigvee_{|I|=n} S^{n\rho_2} \to MU_\mathbb{R}$$

forgetting the group action, refining a basis on $\pi_{2n} MU$. This implies (by the theorem yesterday) that the even slices of $MU_\mathbb{R}$ are pure.

One of the main theorems we will prove is that the odd slices of $MU_\mathbb{R}$ are zero, hence $MU_\mathbb{R}$ is pure.

1.24 DATE: Nov 4th, 2011

$MU_\mathbb{R} = \operatorname{colim} S^{-n\rho_2} \wedge MU_\mathbb{R}(n)$, where $MU_\mathbb{R}(n)$ is the Thom complex (spectrum) of the universal bundle over $BU(n)$.

What is $BU_\mathbb{R}(n)$? It is the limit of the Grassmanian of complex n-planes $Gr_n(\mathbb{C}^N)$ where $\mathbb{Z}/2$ acts by complex conjugation. The fixed point space $BU_\mathbb{R}(n)^{\mathbb{Z}/2} = BO(n)$, where $BO(n)$ is the colimit of real Grassmanians. If X is a space with a $\mathbb{Z}/2$-action, then $[X, BU_\mathbb{R}(n)]^{\mathbb{Z}/2}$ is the set of isomorphism classes of *Real* vector bundles of dimension n on X. What is this? It is a complex vector bundle V of dimension n on X with a $\mathbb{Z}/2$-action on V, such that $V \to X$ is equivariant. $0 \neq \tau \in \mathbb{Z}/2$. This means that we have an isomorphism

$$\tau : V_x \cong V_{\tau x}$$

We further require that this isomorphism is conjugate linear.

Example 1.24.1. If $\mathbb{Z}/2$ acts trivially on X, then a Real vector bundle V on X is $V \cong V_0 \otimes \mathbb{C}$ where V_0 is the fixed point $\mathbb{R}$-vector bundle. ◇

Example 1.24.2. An algebraic variety X over $\mathbb{R}$ with vector bundle. Then look at complex points. ◇

Now Thom spectra have two roles in homotopy theory. They are universal for Thom isomorphisms, and there homotopy groups are certain bordism groups. Look at $MU_{\mathbb{R}}$. let $V \to X$ be a Real vector bundle (of $\dim_{\mathbb{C}} V = n$). We have $X \to BU_{\mathbb{R}}(n)$. We get,

$$Thom(X;V) \to MU_R(n) \to S^{n\rho_2} \wedge MU_{\mathbb{R}}$$

This gives $u \in MU_n^{n\rho_2}(Thom(X;V))$. The *Thom isomorphism*: multiplication by u is an isomorphism

$$MU_{\mathbb{R}}^{\star}(X) \cong MU_{\mathbb{R}}^{\star + n\rho_2}(Thom(X;V)).$$

Suppose that M is an n-dimensional real manifold, i.e. M has a $\mathbb{Z}/2$-action, and TM is a Real vector bundle.

The Pontryagin-Thom construction; We choose $M \subseteq \mathbb{C}^N$, equivariant with normal bundle ν. Then we get

$$\overline{\mathbb{C}}^N \searrow S^{N\rho_2} \to Thom(M;\nu) \to S^{(N-n)\rho_2} \wedge MU_{\mathbb{R}}$$

which gives $S^{n\rho_2} \to MU_{\mathbb{R}}$. We denote this $[M] \in \pi_{n\rho_2} MU_{\mathbb{R}}$. More generally $\pi_{n\rho_2} MU_{\mathbb{R}}$ is cobordism classes of n-dimensional real manifolds.

This is subtle as the usual proof need transversality, but transversality doesn't work well. For example the origin in the real line with its $\mathbb{Z}/2$ flipping action is a fixed point. It can't be equivariantly made transeverse. But it can if there are extra fixed point directions. In the cobordism theory, we are right on the cusp of whether there is enough fixed directions. Cobordisms need to be defined as: A map $M \to \mathbb{C}$ of Real manifolds, transverse at $1, -1$. It is a cobordism between M_{-1} and M_1.

Real oriented cohomology theories, E.

The following are equivalent

- $\overline{x} \in \tilde{E}^{\rho_2}(\mathbb{CP}^\infty)$ restricting to 1;
- A homotopy multiplicative map $MU_{\mathbb{R}} \to E$;
- A Thom isomorphism $E^\star(X) \cong E^{\star + n\rho_2}(Thom(X,V))$ for all Real vector bundles, which is multiplicative in the sense that if $V^n \to X$ and $W^m \to Y$, then

$$\begin{tikzcd}
E^\star(X) \otimes E^\star(Y) \arrow[r] \arrow[d] & E^\star(Thom(X,V) \otimes E^\star(Y,W)) \arrow[d, "\text{mult}"] \\
E^\star(X \times Y) \arrow[r, "\text{Thom}"'] & E^\star(Thom(X \times Y, V \times W))
\end{tikzcd}$$

A Real orientation of E gives a formal group law over $\pi_\star E$. In fact it is defined over $\oplus \pi_{n\rho_2} E$. π is overloaded:

- $\underline{\pi}_* E$ is a Mackey functor.
- $\pi_n^G E = [S^n, E]^G = \underline{\pi}_n E(pt)$.
- $\pi_{n\rho_2} = [S^{n\rho_2}, E]^G$.
- $\pi_n^U X = \underline{\pi}_n X(G) = \pi_n(i^* X)$, where $i^* X$ is the underlying non-equivariant spectrum.

1.24.1 $RO(G)$-graded commutativity

Suppose that E is homotopy commutative. $x : S^v \to X$, and $y : S^W \to E$, then the product is $x \cup y : S^V \wedge S^W \to E \wedge E \to E$. The commutativity of the multiplication tells me that the following diagram commutes (up to homotopy)

$$\begin{tikzcd} S^V \wedge S^W \arrow[d] \arrow[dr] & \\ S^W \wedge S^V \arrow[r] & E \end{tikzcd}$$

where the map $S^V \wedge S^W \to S^W \wedge S^V$ is (non-canonically) an element in $A(G)^\times$. In the usual case this is just an element of ± 1, and this is the usual sign issue.

In our cases of interest, $\pi_0 E = \underline{\mathbb{Z}}$. (True for $E = MU_{\mathbb{R}}$). So the equivariant commutativity just introduces the usual signs. A real orientation $x \in \tilde{E}^{\rho_2}(\mathbb{CP}^\infty)$ gives a formal group law over $\oplus_n \pi_{n\rho_2} E$, and if E is homotopy commutative, then this is a commutative ring.

This maps,

$$\oplus_n \pi_{n\rho_2} E \to \oplus_n \pi^U_{2n} E$$

If x maps to an ordinary complex orientation of $i^* E$, then we have a lift of the formal group law on $\oplus_n \pi^U_{2n} E$.

$$\begin{tikzcd} L \arrow[d, dashed] \arrow[dr] & \\ \oplus_n \pi_{n\rho_2} MU_{\mathbb{R}} \arrow[r] & \oplus_n \pi^U_{2n} MU_{\mathbb{R}} \end{tikzcd}$$

Theorem 1.24.3 (Hu-Kriz, Araki)**.**

Equivalently, every $x \in \pi_{2n} MU = \pi^u_{2n} MU_{\mathbb{R}}$ lifts (refines) to an element $x \in \pi_{n\rho_2} MU_{\mathbb{R}}$. Every stably almost complex manifold is cobordant to a Real manifold. (We already knew this by explicit generators due to Milnor).

...

Now G is an arbitrary group.

Definition 1.24.4. Suppose X is a G-spectrum. A *refinement of homotopy* of X is a map $\hat{W} \to X$, where $\hat{W}$ is a wedge of slice cells, and where non-equivariantly the $S^{n\alpha} X$ form a basis for $\pi^U_* X$. $\diamond$

For this to be possible, the homotopy of $i^* X$ has to be free abelian.

1.25 DATE: Nov 14, 2011

[[★★★ Missed Lectures Nov 7,9, and 11. Anyone take notes?]]

No class Friday.

In the last lecture we were looking at fixed points

claim; The generators $r_1, \cdots, \in L$ ($r_i \in L_{2i}$) whose images $\overline{r_i} \in \pi_{i\rho_2} MU_{\mathbb{R}}$ have the property that $\Phi^{C_2} \overline{r}_i = h_i \in \pi_i MO$ have the property that $h_{2^j - 1} = 0$ $\{h_k | k \neq 2^j - 1\}$.

$\pi_* MU$, Calculated by Thom. There is an equalizer diagram

$$\pi_* MO \to \pi_* H\mathbb{Z}/2 \wedge MO \rightrightarrows \pi_*(H\mathbb{Z}/2 \wedge H\mathbb{Z}/2 \wedge MO)$$

$\mathbb{RP}^\infty = MO(1) = Th(BO(1))$, $MO = \operatorname{colim} \Sigma^{-n} MO(n)$. So we get $\alpha \in MO^1(\mathbb{RP}^\infty)$. The same arguments from last week apply and

$$MO^*(\mathbb{RP}^\infty) = \pi_* MO[[a]]$$

$$MO^*(\mathbb{RP}^\infty \times \mathbb{RP}^\infty) = \pi_* MO[[a, b]]$$

and the map $\mathbb{RP}^\infty \times \mathbb{RP}^\infty \to \mathbb{RP}^\infty$ induces a formal group law over $\pi_* MO$. Now the composite

$$\mathbb{RP}^\infty \xrightarrow{\Delta} \mathbb{RP}^\infty \times \mathbb{RP}^\infty \to \mathbb{RP}^\infty$$

(mult by 2) is null-homotopic. Thus $F(a, a) = a +_F a = 0$.

Theorem 1.25.1 (Quillen)**.** *The formal group law for MO is the unversal one satisfying $F(a, a) = a +_F a = 0$.*

$\pi_* H\mathbb{Z}/2 \wedge MO = H_*(MO; \mathbb{Z}/2)$. There are two ways to think about this. Let $H_* =$ homology with $\mathbb{Z}/2$-coefficients. Then $H_*(MO) = Sym(H_*(\mathbb{RP}^\infty))/b_0 = 1$, where b_i dual to a^i. Another way to think about this is the go through the theory of formal groups using $\mathbb{RP}^\infty$ instead of $\mathbb{CP}^\infty$. Let E be a cohomology theory. Given $\alpha \in \tilde{E}^1(\mathbb{RP}^\infty)$ which restricts to $1 \in E^0(pt)$. Then $E^*(\mathbb{RP}^\infty) = \pi_* E[[\alpha]]$ and a formal group law.

In our particular case $E = H\mathbb{Z}/2 \wedge MO$, we have two classes:

$$a : \mathbb{RP}^\infty \to \Sigma MO \to \Sigma H\mathbb{Z}/2 \wedge MO$$

$$x : \mathbb{RP}^\infty \to \Sigma H\mathbb{Z}/2 \to \Sigma H\mathbb{Z}/2 \wedge MO$$

Thus $E^* \mathbb{RP}^\infty = \pi_* E[[x]]$ with $x +_F y = x + y$. Also $E^* \mathbb{RP}^\infty = \pi_* E[[a]]$, with the formal sum given from the one in MO. But these are isomorphic. Thus....

$$x = \ell(a) = \sum b_n a^{n+1} = a + b_1 a^2 + \dots$$

Where $b_k \in H_k MO$. An exercise shows these are the same b_i from before. ℓ is an isomorphism from the a-formal group to the additive group.

We've shown that $H_* MO$ represents the universal isomorphism of the formal group law over $\pi_* MO$ to the additive formal group law.

What is $\pi_* H\mathbb{Z}/2 \wedge H\mathbb{Z}/2 \wedge MO = A_* \otimes H_* MO$, where A_* is the dual Steenrod algebra. In Milnor's work: There is an action $A \otimes H^*(X) \to H^*(X)$, can be rewritten

$$H^*(X) \to A_* \otimes H^*(X).$$

When $X = \mathbb{RP}^\infty$, then $H^*(\mathbb{RP}^\infty) = \mathbb{Z}/2[x]$, and we get the map

$$x \mapsto \sum f_k x^{k+1} = f(x)$$

we get $f(x + y) = f(x) + f(y)$, so $f(x) = x + \sum \zeta_n x^{2^n}$, so $f_k = 0$ for $k \neq 2^n - 1$. $\zeta_n \in A_*$. Thus (using Steenrod algebra relations) $A_* = \mathbb{Z}/2[\zeta_1, \zeta_2, \dots]$.

Atiyah-Hirzebruch interpreted Milnor's computation as $\mathcal{S}pec\, A_* = Aut(\mathbb{G}_a)$. This was the beginning of the entry of formal groups into algebraic topology and homotopy theory.

$\pi_* MO \to H\mathbb{Z}/2_* MO \rightrightarrows A_* \otimes H\mathbb{Z}/2_* MO$. The first is the Scheme of formal group laws F with an isomorphism to $\mathbb{G}_a$, and the second consists of those F with an isomorphism of F with $\mathbb{G}_a$, and an additional automorphism of $\mathbb{G}_a$. The pair of maps are the obvious ones.

Now $\ell(a) = \sum b_k a^{k+1}$. $\ell^{(2)} = \sum b_{2^j-1} a^{2^j}$. Then $(\ell^{(2)})^{-1} \circ \ell(a) = \sum h_i a^i$ (and $h_{2^j-1} = 0$). (This is easy and an exercise for the reader). The conclusion is that $\pi_* MO = \mathbb{Z}/2\{h_i \mid i \neq 2^k - 1\}$.

Now let's do the MU analog. We can do this at any prime, but we're going to assume that everything is localized at $p = 2$.

$$\pi_* MU \to H_* MU$$

$x \in H^2(\mathbb{CP}^\infty; \mathbb{Z})$, the generator, and we have $z \in MU^2(\mathbb{CP}^\infty)$. A similar discussion holds. The homology of MU is the symmetric algebra of $H_* \mathbb{CP}^\infty / m_0 = 1$. It is generated by the m_i which are dual to z^i. $\mathcal{S}pec\, H_* MU = Iso(F^{MU}, \mathbb{G}_a)$ and $x = \ell(z) = z + \sum m_n z^{n+1}$.

Notation: $Q_{2n} = \pi_{2n} MU / I^2$ which $I = \oplus_{j>0} \pi_{2j} MU$. Let $IH = \oplus_{j>0} \mathbb{H}_{2j} MU$, and $QH_* = IH/IH^2$. Then $QH_{2n} = \mathbb{Z}_{(2)}\{m_n\}$.

Theorem 1.25.2 (Milnor). *$Q_{2n} \to QH_{2n}$ is an isomorphism if $n \neq 2^j - 1$, and $Q_{2(2^j-1)} \hookrightarrow QH_{2n} \to \mathbb{Z}/2$ (short exact sequence).*

$\ell(z) - \sum m_n Z^{n+1}$, and let $\ell^{(2)}(z) = \sum m_{2^j-1} z^{2^j}$.

Theorem 1.25.3 (Cartier). *$(\ell^{(2)})^{-1} \circ \ell$ has coefficients in $\pi_* MU$ (i.e. over L and not just $L \otimes Q$).*

Example 1.25.4. Consider the multiplicative formal group, $x +_F y = 1 - (1-x)(1-y) = x + y - xy$. This $\log(1-x) = x + \frac{x^2}{2} + \cdots$ gives an isomorphism of this with the additive group. $\ell^{(2)}(x) = \sum \frac{x^{2^n}}{2^n}$. Then

$$(\ell^{(2)})^{-1} \circ \ell(x)$$

has integer coefficients is equivalent to $\exp(\sum \frac{x^{2^n}}{2^n}) \in \mathbb{Z}_{(2)}[[x]]$, which is the Artin Hasse exponential formula. ◇

Using Cartier's theorem, let's define $r_i \in \pi_{2iMU}$ (over $\mathbb{Z}_{(2)}$) by

$$(\ell^{(2)})^{-1} \circ \ell(z) = \sum r_i z^{i+1}$$

The easy to check $r_i \equiv m_i \bmod IH^2$ if $i \neq 2^j - 1$, and that $r_{2^j-1} \equiv 2m_{2^j-1}$. So the r_i are the algebra generators. $\pi_* MU = \mathbb{Z}_{(2)}[r_1, \dots]$. This gives elements $\overline{r}_i \in \pi_{i\rho_2} MU_{\mathbb{R}}$. The claim is that these are the elements such that

$$\Phi^{C_2}(\overline{r}_i) = h_i$$

$(h_{2^j-1} = 0)$.

Let's try to relate real bordism to unoriented bordism. $\overline{x} : \mathbb{CP}^\infty \to S^{\rho_2} \wedge MU_{\mathbb{R}}$ which has the spacial property that the restriction to $\mathbb{CP}^1$ is the usual thing. Now apply Φ^{C_2}. We get the class $a : \mathbb{RP}^\infty \to S^1 \wedge MO$. The map,

$$\pi_{*\rho_2} X \to \pi_* \Phi^{C_2} X$$

is additive and multiplicative. Thus if we define $\overline{m}_n \in \pi_{n\rho_2} H\underline{\mathbb{Z}} \wedge MU_{\mathbb{R}}$, we have $\overline{\ell}(\overline{x}) = \sum \overline{m}_n \overline{x}^{n+1}$, and similarly for $\overline{\ell}^{(2)}$. Applyng Φ^{C_2} gives the ℓ and $\ell^{(2)}$ from before.

The result now follows by applying Φ^{C_2} to $(\overline{\ell}^{(2)})^{-1} \circ \overline{\ell}(\overline{x}) = \sum \overline{r}_n \overline{x}^{n+1}$

1.26 DATE: Nov 16, 2011

In the last class we defined these generators r_i for the Lazard ring, and mapped this to $\oplus \pi_{n\rho_2} MU_{\mathbb{R}}$ sending r_i to $\overline{r}_i$. We defined $h_i = \Phi^{C_2} \overline{r}_i \in \pi_i MO$, and we saw $h_{2^j-1} = 0$, and $\pi_* MO = \mathbb{Z}/2[h_i \mid i \neq 2^j - 1]$.

For $G = C_{2^n}$, we can generalize to $MU^{(G)} = N_{C_2}^G MU_{\mathbb{R}}$. The C_2-spectrum underlying $MU^{(C_4)} = MU_{\mathbb{R}} \wedge MU_{\mathbb{R}}$ where the C_4-action sends $(a, b) \mapsto (\overline{b}, a)$. The proof from last time generalizes.

γ a generator of G. We know $\pi_*^u MU^{(G)}$, because of the yoga of complex oriented cohomology theories, but the same techniques from last time also apply. We learn that

$$\pi_*^u MU^{(G)} = \mathbb{Z}_{(2)}[\underbrace{r_1, \gamma r_1, \cdots, \gamma^c r_1}_{|G|/2 \text{ times}}, r_2, \gamma r_2, \cdots, \gamma^c r_2, \cdots]$$

where $c = |G|/2 - 1$. Then $\gamma^{\frac{|G|}{2}} r_1 = -r_1 \Leftrightarrow$ complex conjugation. If we let $G \cdot r_i =$ the collection of $r_i, \gamma r_i, \cdots, \gamma^c r_i$. So we say,

$$\pi_*^u MU^{(G)} = \mathbb{Z}_{(2)}[G \cdot r_1, G \cdot r_2, \cdots]$$

Warning: The r_i for G depend on G.

Construction of the r_i for $G = C_4$. $\pi_*^u MU^{(C_4)} = \pi_* MU \wedge MU$. This represents a pair of formal group laws and an isomorphism between them. The two formal group laws are F and F^γ, and we have an isomorphism from F to F^γ. With this description we can work out the whole C_4-action.

$$\begin{tikzcd} F \arrow[r] \arrow[d, "\ell"'] & F^\gamma \arrow[d, "\ell^\gamma"] \\ \mathbb{G}_a & \mathbb{G}_a \end{tikzcd}$$

where $\ell(x) = \sum m_n x^{n+1}$, and $\ell^\gamma(x) = \sum m_n^\gamma x^{n+1}$. We also have $(\ell^\gamma)^{(2)} = \sum m_{2^n-1}^\gamma x^{2^n}$ [???]

Claim:

$$((\ell^\gamma)^{(2)})^{-1} \circ \ell = \sum r_i x^{i+1}$$

has coefficients in $\pi_* MU \wedge MU$. This is because $(\ell^\gamma)^{-1} \circ \ell$ has coefficients in $\pi_* MU \wedge MU$. $((\ell^\gamma)^{(2)})^{-1} \circ \ell^\gamma$ has coefficients in $\pi_* MU \wedge MU$ by Cartier's result.

The argument from last time that showed that the r_i are generators generalizes to this case as well.

This all comes from manipulations of formal groups. It actually can be made to take place in $\oplus_i \pi_{i\rho_2}^{C_2} MU^{(G)}$. The consequence of that is that we get a map

$$\mathbb{Z}_{(2)}[G \cdot r_1, G \cdot r_2, \dots] \to \oplus_i \pi_{i\rho_2}^{C_2} MU^{(G)}$$
$$\gamma^j r_i \mapsto \gamma^j \overline{r}_i$$

When $G = C_8$, this is the spectrum and its homotopy groups are what we will want to understand.

The Main Theorem is that $MU^{(G)}$ is pure. Let's sketch the approach to proving this.

[[★★★ Q: Isn't this just a formal consequence of the fact that $MU^{(G)}$ is a smash of $MU_{\mathbb{R}}$s? Ans: No! For example $N_{C_2}^{C_4} H\underline{\mathbb{Z}} \neq H\underline{\mathbb{Z}}$ nor is it even pure. It is a crappy spectrum.]]

We would like to define a G-spectrum J_n such that $MU^{(G)}/J_n$ is $\pi_* MU^{(G)}/$ the ideal of all elements of degree $\geq n$. Then form a tower,

$$\cdots \to MU^{(G)}/J_2 \to MU^{(G)}/J_1 = H'$$

[H' is written $R(\infty)$ in the paper]. We then find that the fiber of $MU^{(G)}/J_n \to MU^{(G)}/J_{n-1}$is contractible when n is odd and is $H' \wedge$ a wedge of slice cells of dimension n, when n is even. Proving $H' = H\mathbb{Z}$ then implies that $MU^{(G)}$ is pure and that $\{MU^{(G)}/J_n\}$ is the slice tower.

We construct MU/J_n using the *method of polynomial algebras*. For each $\overline{r}_i$ choose a C_2-equivariant map

$$\overline{r}_i : S^{i\rho_2} \to MU^{(G)}$$

representing it. Then extend to a map of associative algebras

$$T(S^{i\rho_2}) \to MU^{(G)}$$

Here the free algebra $T(S^{i\rho_2}) = \vee_{n \geq 0} S^{in\rho_2}$. This map is C_2-equivariant. Now we apply $N_{C_2}^G$ and we get

$$\begin{tikzcd} N_{C_2}^G T(S^{i\rho_2}) \arrow[r] & N_{C_2}^G MU^{(G)} \arrow[d] \\ & MU^{(G)} \end{tikzcd}$$

Now by the formula for the norm of a wedge this first term is isomorphic to $S^0[G \cdot \overline{r}_i]$.

Example 1.26.1. $G = C_4$. Then $S^0[G \cdot \overline{r}_1]$ is $S^0[\overline{r}_1] \wedge S^0[\gamma \overline{r}_1]$ which is equal to

$$(S^0 \vee S^{\rho_2} \vee S^{2\rho_2} \vee \cdots) \wedge (S^0 \vee S^{\rho_2} \vee S^{2\rho_2} \vee \cdots)$$

which is just,

$$S^0 \vee \underbrace{(S^{\rho_2} \vee S^{\rho_2})}_{C_{4+} \wedge_{C_2} S^{\rho_2}} \vee (\underbrace{S^{2\rho_2} \vee S^{2\rho_2}}_{C_{4+} \wedge_{C_2} S^{2\rho_2}} \vee \underbrace{S^{\rho_2} \wedge S^{\rho_2}}_{S^{\rho_4} = N_{C_2}^{C_4} S^{\rho_2}}) \vee \cdots$$

◇

More generally $H_*^u S^0[G \cdot \overline{r}_i] = \mathbb{Z}[r_i, \gamma r_i, \cdots, \gamma^c r_i]$, and $S^0[G \cdot \overline{r}_i]$ is a wedge of slice cells of the from $G_+ \wedge_H S^{m\rho_H}$ (not desuspensions). Now we take this

$$S^0[G \cdot \overline{r}_i] \to MU^{(G)}$$

and smash together and take a colimit to get a G-equivariant map

$$S^0[G \cdot \overline{r}_1, G \cdot \overline{r}_2, \dots] \to MU^{(G)}$$

Then M_n is the sub-wedge of slice cells of $S^0[G \cdot \overline{r}_1, G \cdot \overline{r}_2, \dots]$ of cells of dimension $\geq n$. Then $J_n = MU^{(G)} \wedge_{S^0[G \cdot \overline{r}_1, \dots]} M_n$. We get the desired property for $MU^{(G)}/J_n$.