The Classification of Two-Dimensional Extended Topological Field Theories

by

Christopher Schommer-Pries

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Committee in charge:
Professor Peter Teichner, Chair
Professor Alan Weinstein
Professor Ori Ganor

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The dissertation of Christopher Schommer-Pries is approved:

Chair Date

Date

Date

University of California, Berkeley

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Abstract

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We provide a complete generators and relations presentation of the 2-dimensional extended unoriented and oriented bordism bicategories as symmetric monoidal bicategories. Thereby we classify these types of 2-dimensional extended topological field theories with arbitrary target bicategory. As an immediate corollary we obtain a concrete classification when the target is the symmetric monoidal bicategory of algebras, bimodules, and intertwiners over a fixed commutative ground ring R. In the oriented case, such an extended topological field theory is equivalent to specifying a separable symmetric Frobenius algebra.

Along the way we collect together the notion of symmetric monoidal bicategory and define a precise notion of generators and relations for symmetric monoidal bicategories. Given generators and relations, we prove and abstract existence theorem for a symmetric monoidal bicategory which satisfies a universal property with respect to this data. We also prove a theorem which provides a simple list of criteria for determining when a morphism of symmetric monoidal bicategories is an equivalence. We introduce the symmetric monoidal bicategory of bordisms with structure, where the allowed structures are essentially any structures which have a suitable sheaf or stack gluing property.

We modify the techniques used in the proof of Cerf theory and the classification of small codimension singularities to obtain a bicategorical decomposition theorem for surfaces. Moreover these techniques produce a finite list of local relations which are sufficient to pass between any two decompositions. We deliberately avoid the use of the classification of surfaces, and consequently our techniques should be readily adaptable to higher

dimensions. Although constructed for the unoriented case, our decomposition theorem is engineered to generalize to the case of bordisms with structure. We demonstrate this for the case of bordisms with orientations, which leads to a similar classification theorem.

Professor Peter Teichner
Dissertation Committee Chair

To Natalie Schommer-Pries,

for her loving support and constant encouragement.
Without her, this would not have been possible.

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Above all, I wish to thank my advisor, Peter Teichner, for his years of support and guidance. Without his help this dissertation would not have been possible. I felt that he has always provided an atmosphere of encouragement in which I was free to explore whatever mathematical ideas I might fancy. I would especially like to thank Peter who, after hearing an earlier attempt at the main theorems of this work, suggested that I learn Cerf theory. That crucial piece of advice set in motion a snowball which has grown into this dissertation.

I would also like to extend my gratitude to Stephan Stolz. It was a conversation with him which initially sparked my serious interest in the 2-dimensional bordism bicategory. Thinking that 2-dimensional extended topological theories were fairly easy, I was certain that the value associated to the circle should be the center of the algebra associated to the point. Stephan pointed out to me that, on the contrary, it should be the universal trace space. It turns out we were both right, as Lemma 4.6.11 now shows. I would also like to thank him for our numerous conversations since that time. His insights have always been extremely helpful.

I would also like to give special acknowledgment to Chris Douglas. Our conversations, which have grown into a collaboration, have had a tremendous impact on how I think about these results. In particular they have helped shaped my view that the importance of this work rests in the potential future applications and developments.

Chapter 1

Introduction

1.1 Introduction

Recently there have been many exciting developments in the fields of topology, quantum field theory, and higher categories. This dissertation solves a particular problem in the intersection of these three fields, the classification of extended 2-dimensional topological field theories, and has the potential for many future applications and developments. A *topological quantum field theory* (TQFT), as axiomatized by Atiyah [Ati88], is a functor between two particular categories, a geometric category and an algebraic category.

The geometric category is a category of *bordisms*. There is one such category for each non-negative integer, d, leading to a notion of d-dimensional TQFT. The d-dimensional bordism category has objects which are closed $(d-1)$-dimensional manifolds and morphisms which are d-dimensional bordisms between these, i.e. a compact d-dimensional manifold whose boundary is divided into two components, the source and the target. These bordisms are taken up to diffeomorphism, relative to the boundary, and composition is given by gluing.

The algebraic category is usually the category of vector spaces over a fixed ground field. Both of these categories have extra structure making them into symmetric monoidal categories and a topological quantum field theory is required to be a symmetric monoidal functor. The symmetric monoidal structure on the bordism category is given by the disjoint union of manifolds, and the symmetric monoidal structure on vector spaces is given by tensor product. Thus a TQFT, Z, comes equipped with canonical isomorphisms $Z(M_1 \sqcup M_2) \cong Z(M_1) \otimes Z(M_2)$.

There are many reasons to study topological quantum field theories, but one reason

is that they exhibit a beautiful relationship between algebra and geometry or topology. It is not surprising that a TQFT provides some relations between the two, since it is a functor between an algebraic and a geometric category, but what is surprising is how seemingly unrelated, yet often well known algebraic structures emerge via TQFTs. Probably the best known example of this is the following theorem.

Theorem 1.1.1. *(Folklore) The 2-dimensionsal oriented bordism category is the free symmetric monoidal category with a single commutative Frobenius algebra object. In particular the category of oriented 2-dimensional TQFTs is equivalent to the category of commutative Frobenius k-algebras.*

The history of this theorem is somewhat intricate. The first statement that 2D TQFTs are commutative Frobenius algebras appears in Dijkgraaf's thesis [Dij89], but is also listed as a folk theorem by Voronov [Vor94]. This was surely known to Atiyah and Segal and many others, but mathematical proofs didn't appear until the work of L. Abrams [Abr96], S. Swain [Saw95], F. Quinn [Qui95], and B. Dubrovin [Dub96].

This theorem is really two theorems packaged into one. First it is a theorem which identifies the bordism category in terms of generators and relations. In particular the 2-dimensional bordism category is equivalent to the abstract symmetric monoidal category generated by a single object, S^1, and the morphism depicted in Figure 1.1, together with

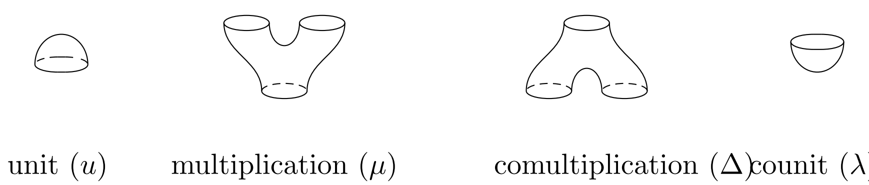


Figure 1.1: 2D TQFTs as Commutative Frobenius Algebras

the obvious relations coming from Morse handle slides and birth/death cancelation of Morse critical points. The second half of this theorem then identifies these generator and relations with familiar algebraic concepts. Thus a 2-dimensional oriented TQFT is completely determined by the vector space $Z(S^1)$ and the linear maps obtained by applying the TQFT to

each of the generators. These give linear maps:

$$\begin{aligned} u &: Z(\emptyset) = k \to Z(S^1) \\ \mu &: Z(S^1) \otimes Z(S^1) \to Z(S^1) \\ \Delta &: Z(S^1) \to Z(S^1) \otimes Z(S^1) \\ \lambda &: Z(S^1) \to Z(\emptyset) = k. \end{aligned}$$

The relations, which are derived from purely topological facts about the bordism category, ensure that these maps give $V = Z(S^1)$ the structure of a commutative Frobenius algebra.

An *extended* topological field theory is a higher categorical version of a TQFT. The symmetric monoidal bordism category is replaced by a symmetric monoidal bordism n-category and an extended topological field theory is a symmetric monoidal n-functor from this to a, usually algebraic, target symmetric monoidal n-category.

The simplest example of this is the 2-dimensional bordism bicategory. The objects of this bicategory are (compact) 0-dimensional manifolds, the 1-morphisms are 1-dimensional bordisms between these and the 2-morphisms are 2-dimensional bordisms between the bordisms, taken up to diffeomorphism rel. boundary.

In this dissertation we provide a generators and relations presentation of the 2-dimensional bordism bicategory as a symmetric monoidal bicategory. This classifies 2-dimensional topological field theories with any target symmetric monoidal bicategory in terms of a small amount of very explicit data. More important than this theorem itself are the methods of proof and the supporting theorems and lemmas. Part of these are foundational and all are designed with an eye towards generalizations. The structure of this dissertation has been divided into three parts. We have made an effort separate out those aspects which are purely algebraic/categorical from those which are purely geometric/topological and this is reflected in the three chapters of the sequel.

Chapter 2 is written entirely in the language of differential topology and makes no mention whatsoever of symmetric monoidal bicategories or the bordism bicategory, although its main results will be applied in this setting. Chapter 3 is purely algebraic/categorical and deals with some foundational material on symmetric monoidal bicategories. For example the definition of symmetric monoidal bicategory was scattered in several pieces throughout the literature, and there were some very minor gaps.

We fill in these gaps and prove two general theorems about symmetric monoidal

bicategories. One of these provides a precise notion of generators and relations for symmetric monoidal bicategories. Given certain generators and relations data, we prove the existence of an abstract symmetric monoidal bicategory which is presented by that data. This symmetric monoidal bicategory satisfies a universal property making it easy to understand symmetric monoidal functors out of it. In particular this gives a precise meaning to the main theorems of this dissertation. Our generators and relations presentation gives an abstract symmetric monoidal bicategory which, by the universal property, comes equipped with a symmetric monoidal functor to the bordism bicategory. The main theorem proves this is an equivalence of symmetric monoidal bicategories. To aid this proof, we proof another algebraic theorem in Chapter 3 which provides three relatively easy to check criteria which exactly characterize those symmetric monoidal functors which are symmetric monoidal equivalences.

In Chapter 4 we unite the categorical and the geometric results. We define the symmetric monoidal bordism bicategory and several variations of it. We then prove the main theorems in the unoriented and oriented cases. By the results of Chapter 3 , there are three criteria which must be checked to prove these classification theorems. The first two are relatively easy. The last criteria follows from the main differential topology results of Chapter 2. Finally, we discuss some of the consequence of these theorems. In particular we completely analyze 2D topological field theories with values in the bicategory of algebras, bimodules and intertwiners (over a fixed commutative ground ring). We identify these in terms of classically studied structures.

1.2 Symmetric Monoidal Bicategories

The mathematical axiomatization of symmetric monoidal bicategories goes back to the work of Kapranov and Voevodsky [KV94a, KV94b] on braided monoidal 2-categories. They proposed a definition of braiding for strictly associative monoidal 2-categories. There was a minor omission in this original definition of braided (strict) monoidal 2-category, which was repaired in the work of Baez and Neuchl [BN96], where the definition is also simplified and put into a more conceptual context.

It was further clarified in the work of Day and Street [DS97]. They explain how the categorification of monoidal, braided monoidal, and symmetric monoidal categories gains an additional layer. There are monoidal 2-categories, braided monoidal 2-categories, *sylleptic* monoidal 2-cateogries and, finally, symmetric monoidal 2-categories. Just as symmet-

ric monoidal categories are braided monoidal cateogries which satisfy additional axioms, symmetric monoidal bicategories are sylleptic monoidal bicategories satisfying additional axioms. This is in contrast to the relationship between braided and sylleptic monoidal bicategories: a sylleptic monoidal bicategory not only satisfies additional axioms compared to a braided monoidal bicategory, but it is equipped with additional structure, a *syllepsis*. Again, all this is carried out using a partially strict notion of monoidal 2-category called a *Gray monoid*. The authors justify this by invoking the coherence theorem of Gordon, Powers and Street.

Gordon, Powers and Street, in [GPS95], introduced what is essentially a fully weak notion of tricategory and accompanying notions of trihomomorphism, tritransformation, and higher morphisms. Mirroring the well know definition of a (weak) monoidal category as a bicategory with one object, they define a monoidal bicategory to be a tricategory with one object. This is the most common form encountered in examples, and it is essentially the definition we use below. These authors also prove a coherence theorem, which ensures that any such monoidal bicategory is equivalent to a Gray monoid.

Day and Street consequently defined braided, sylleptic and symmetric monoidal bicategories only in the context of Gray monoids. To quote them,

> Examples naturally occur as monoidal bicategories rather than Gray monoids. However, the coherence theorem of [GPS95] allows us to transfer our definitions and results. ...

This transference was carried out explicitly for braided and sylleptic monoidal bicategories in part of the thesis work of P. McCrudden and occurs in the appendices of [McC00]. The symmetric monoidal bicategories of primary interest to the current work are, naturally, of this fully weak kind.

There are two remaining gaps, which are relatively straightforward to fill, but which we were unable to find in the literature. We begin Chapter 3 by reviewing the definition of symmetric monoidal bicategories and by filling these gaps. We also introduce the operation of symmetric monoidal *whiskering*, which is then used to construct a tricategory of symmetric monoidal bicategories. These results are relatively straight forward and mirror results existing in the literature.

In addition to the above, we develop two main results about symmetric monoidal bicategories. The first of these is a theorem which characterizes in simple terms precisely those symmetric monoidal functors (also called symmetric monoidal *homomorphisms*) which

are symmetric monoidal equivalences. We call this "Whitehead's theorem for symmetric monoidal bicategories" (Theorem 3.4.10) because it precisely mirrors Whitehead's theorem in topology.

Whitehead's Theorem states that a map between reasonable topological spaces is a homotopy equivalence if and only if it induces an isomorphism of all homotopy groups and all base points. There is a well known relationship between n-categories in which all morphisms are invertible (n-groupoids) and homotopy n-types, i.e. reasonable spaces in which all homotopy groups above n vanish. The theories of these two objects are essentially the same, a statement which has become known as the *homotopy hypothesis*. For small n, this can be made into a precise statement, which can be verified. One consequence of this is that a functor between ordinary 1-categorical groupoids is an equivalence if and only if it is an isomorphism on π_0 and at $\pi_1(x)$ for every object x. A similar statement holds for 2-groupoids, which are bicategories in which all 1-morphisms and 2-morphisms are invertible.

There is an analog of this which is valid in the non-groupoid setting. It is no longer enough to simply check homotopy groups, since we must deal with non-ivertible morphisms. The analog of Whitehead's theorem for ordinary categories states that a functor between categories is an equivalence if it is essentially surjective and fully-faithful. These conditions replace the notion of isomorphism of all homotopy groups. There is a well known bicategorical analog as well. A homomorphism of bicategories is an equivalence if and only if it is essentially surjective on objects, essentially full on 1-morphisms, and fully-faithful on 2-morphisms. Just as groupoids correspond to certain spaces, symmetric monoidal groupouids correspond to certain stable, or E_∞-spaces. There is an analog of Whitehead's theorem valid in this setting as well.

What we prove is that a symmetric monoidal homomorphism is a symmetric monoidal equivalence if and only if it is essentially surjective on objects, essentially full on 1-morphisms, and fully-faithful on 2-morphisms. In otherwords as symmetric monoidal homomorphism is a symmetric monoidal equivalence if and only if it is an equivalence of underlying (non-monoidal) bicategories. In theory, it is possible to prove this directly, using a single application of the axiom of choice. In practice there is too much data to keep track of. Instead we prove Theorem 3.4.10, by proving a series of smaller lemmas and reductions, building up to this theorem piece by piece.

The second main result of Chapter 3 is a precise theory of generators and relations

for symmetric monoidal bicategories. There are many versions of what data constitute "generators," which make the theory of generators and relations more tractable. For example, one might wish to take a bicategory as the generating data and then form the "free symmetric monoidal bicategory" generated by that bicategory. Such a construction exists and is easier then the construction we present. The problem, though, is that in our applications the generating data we wish to give is not easily organized into a bicategory. The most important problem is that we may wish to have certain generating 1-morphisms and 2-morphisms whose sources and targets are *not* generating objects. For example in the 2-dimensional bordism bicategory, every object is (equivalent to) a disjoint union of points, so the single point is the only generating object. However certain generating 1-morphisms, for example the "elbow" $\subset$, have sources which are not a single point, but a disjoint union of points. In other words we wish to allow generating data which has sources and targets which are *consequences* of the previous generating data.

One method of dealing with this is to develop a *relative* version of generators. Given a symmetric monoidal bicategory $(\mathsf{C}, \otimes)$, we may forget the symmetric monoidal structure and just remember the underlying bicategory. This leads to a functor between tricategories $U : \mathsf{SymBicat} \to \mathsf{Bicat}$. This is part of a (weak) adjunction,

$$F : \mathsf{Bicat} \rightleftarrows \mathsf{SymBicat} : U.$$

The symmetric monoidal bicategory $F(\mathsf{B})$ is the "free symmetric monoidal bicategory" generated by B. If we have a map of symmetric monoidal bicategories $\mathsf{C}_0 \to \mathsf{C}_1$, this gives rise to a map of underlying bicategories $U(\mathsf{C}_0) \to U(\mathsf{C}_1)$. Thus there is a functor of slice tricategories, $U : \mathsf{SymBicat}_{\mathsf{C}_0/} \to \mathsf{Bicat}_{/U(\mathsf{C}_0)}$. The relative freely generated category is the (weak) left adjoint to this.

$$F : \mathsf{Bicat}_{/U(\mathsf{C}_0)} \rightleftarrows \mathsf{SymBicat}_{/\mathsf{C}_0} : U.$$

We may now give one definition of the generating data and notion of freely generated bicategory by repeatedly applying this relative functor.

We begin with a discrete bicategory of generating objects, C_0. Then we form the free symmetric monoidal bicategory generated by this and then forget the symmetric monoidal structure. This gives us an ordinary bicategory B_1. The generating 1-morphisms are then part of a enlargement $\mathsf{C}_1 \supset \mathsf{B}_1$. We apply the relative freely functor to C_1, forget to obtain B_2. Lastly, we enlarge B_2 to C_2 by adding the generating 2-morphisms, and then

apply the relative free functor a final time. Thus the generating data consists of the three iterations of bicategories C_0, C_1, and C_2.

To carry out this process as described we would have to supplement the existing literature with a good theory of weak adjunction between tricategories. Instead we may simply describe the end result. Although less conceptual, this is the approach that we take here. We give a precise inductive definition of generating data and prove the existence of a freely generated symmetric monoidal bicategory generated by that data. In the end this gives the same result as the relative construction outlined above, but without needing the theory of weak adjunctions between tricategories. Finally, we discuss how to impose relations on bicategories.

1.3 Topological Field Theories and Planar Decompositions

Modern theoretical physics has had an increasingly influential effect on mathematics over the past three decades. In particular the emphasis of E. Witten's work on topological aspects of quantum field theory has lead to many significant developments in both fields, many of which continue to be influential to this day. It was in part an attempt to better understand the mathematical aspects of some of these developments that led Atiyah and Segal to develop a rigorous mathematical formalism in which to study quantum field theories.

In the late 1980s, Graeme Segal wrote and circulated a manuscript, now published [Seg04], in which he gave a mathematical definition of conformal field theory. He conceptualized it as a functor from a certain category, roughly a category whose objects are 1-manifolds and morphisms are conformal bordisms, to the category of Hilbert spaces and linear operators. At approximately the same time, Atiyah, in his 1988 paper [Ati88], introduced the definition of a *topological* quantum field theory (TQFT). As we mentioned earlier, this is now understood to be equivalent to defining a TQFT as a symmetric monoidal functor $\mathsf{Bord}_d \to \mathsf{Vect}$.

In addition to providing a connection between algebra and geometry, TQFTs have at least two other significant uses. Firstly, they serve as toy models for more complicated non-topological field theories. For example Segal's axiomatic approach to conformal field theory is very closely related to 2-dimensional topological field theories and they have many structures and general features in common. By studying the simpler topological theories,

one might hope to gleam some structure or insight applicable to more general quantum field theories.

In the Folklore Theorem 1.1.1 we saw an example of how we can potentially obtain complete information about the entire category of topological field theories. This is particularly important for understanding the third principle applications of topological quantum field theories: manifold invariants. Indeed, one of Atiyah's original motivations for axiomatizing TQFTs was an attempt to capture the mathematical structure of the 3-manifold invariants coming out of quantum Chern-Simmons theory.

A symmetric monoidal functor must send the monoidal unit of the source category to the monoidal unit of the target category (at least up to canonical isomorphism). The monoidal unit of the bordism category is the empty manifold $\emptyset$, and thus if Z is a TQFT, $Z(\emptyset) \cong k$, the monoidal unit for the category of k-vector spaces. Moreover, any closed d-manifold, W, can be viewed as a bordism from $\emptyset$ to itself and thus $Z(W)$ is an endomorphism of $Z(\emptyset) \cong k$, i.e an element of k, and this element only depends on the diffeomorphism class of W.

Unfortunately these manifold invariants appear to be most interesting in low dimensions. A direct check shows that TQFTs can distinguish the topological type of 0-, 1-, and 2-dimensional manifolds. Work of Freedman-Kitaev-Nayak-Slingerland-Walker-Wang [FKN$^+$05] shows that unitary[1] TQFTs cannot detect smooth structures[2] in dimension $d = 4$. Work of Kreck and Teichner [KT08], builds of this, showing that unitary 5-dimensional TQFTs can detect simply connected 5-manifolds, but that unitary TQFTs of dimension six or greater cannot even detect the homotopy type of manifolds.[3] This leaves dimension three, where we have the following open problem, although work of Calegari, Freedman, and Walker [CFW08], provides evidence that this might be the case.

Open Problem 1.3.1. *Can 3-dimensional TQFTs distinguish closed 3-manifolds?*

One important concept in the study of quantum field theories is the notion of *locality*. Most quantum field theories coming from physics are defined in terms of local quantities

[1] A *unitary* TQFT is a particular kind of oriented TQFT, taking values in complex vector spaces. A TQFT is unitary if $Z(\overline{M}) \cong \overline{Z(M)}$, where $\overline{M}$ denotes M with the opposite orientation.

[2] The "topological field theories" arising from Donaldson and Seiberg-Witten invariants do not contradict this for two reasons. First they are not honest TQFTs in the sense of Atiyah as they are not defined for all bordisms, and moreover even if they were, they would lead to non-unitary TQFTs.

[3] Unitary TQFTs in these dimensions can detect the cohomology of the manifold as an abelian group, but cannot detect the cup product structure.

(functions, sections of bundles, etc.) and hence have a local property not expressed in the Atiyah-Segal axioms. One method of incorporating locality into the Atiyah-Segal axioms, arising out of the work of D. Freed, F. Quinn, K. Walker, J. Baez, and J. Dolan, is to redefine a topological quantum field theory in the language of higher categories.

A category consists of objects and morphisms between the objects. A higher category consists of objects, morphisms between the objects, morphisms between the morphisms (called 2-morphisms), morphisms between the 2-morphisms (called 3-morpisms), and so on. There are various compositions of these morphisms, and various coherence data associated with these various compositions. Making this precise has become an industry in its own right, and there are many competing definitions of (weak) n-category. However, these are all equivalent for $n = 2$ and in particular are equivalent to the notion of *bicategory* introduced by Bénebou in 1967 [Bén67].

One prototypical example of an n-category is the d-dimensional bordism n-category. The objects consist of closed $(d-n)$-manifolds, the 1-morphisms consist of $(d-n+1)$-dimensional bordisms between these, the 2-morphisms consist of $(d-n+2)$-dimensional bordisms between the bordisms, and so on, up to dimension n. Thus an *extended* topological field theory is a functor of n-categories from this bordism n-category to some (usually algebraic) target n-category. More precisely, an extended topological field theory should be a *symmetric monoidal* functor between symmetric monoidal n-categories. Making this precise is, needless to say, extremely laborious.

One reason to be interested in the study of extended topological field theories is that, again, they serve as a toy model for *non-topological* extended quantum field theory. Such extended non-topological QFT has arisen, for example, in the Stolz-Teichner program on elliptic cohomology [ST04, ST08]. Their proposal, over simplifying a bit, is that the space of 2-dimensional extended supersymmetric Euclidean field theories should be the classifying space of the generalized cohomology theory *topological modular forms* (which is the universal elliptic cohomology theory). Their program has had great success in lower dimensions, where they prove that 0-dimensional supersymmetric Euclidean field theories yield de Rham cohomology [HKST08] and 1-dimensional supersymmetric Euclidean field theories produce K-theory. These results have so far depended heavily on being able to identify generators and relations for these more exotic bordism categories. Thus it is desirable to study extended topological field theories in such a way that either generalizes to or gives insight about the extended non-topological setting.

Anther reason to be interested in extended topological field theories that they are inherently more computable. If we are given an extended topological field theory and we want to evaluate it on, say, a d-dimensional manifold, we may start by chopping the manifold into smaller, more elementary pieces. As the category number increases, these elementary pieces become simpler and fewer in number. In principle, the topological field theory becomes easier to compute.

Finally, an important appearance of extended TQFTs is in the construction of examples of TQFTS. Many known 3-dimensional TQFTs arise out of a method known as the Reshetikhin-Turaev construction. This construction takes as input a certain algebraic gadget called a *modular tensor category*. This is a particular kind of braided monoidal category equipped with additional structure. Given a modular tensor category, M, the Reshetikhin-Turaev construction provides a recipe for constructing a 3-dimensional TQFT. However the details of this construction actually produce an *extended* TQFT. In addition to assigning data to 3-manifolds and surface, data is assigned to 1-manifolds as well. Specifically the circle is assigned the category M. Surfaces with boundary are then assigned linear functors, for example the second bordism of Figure 1.1 is assigned the functor,

$$\otimes : M \times M \to M.$$

This means that most examples of 3D topological quantum field theories are actually extended at least to circles. For example, this is true of quantum Chern-Simmons theory. This begs the following open question, although see [FHLT09] for some recent progress.

Open Problem 1.3.2. *Which 3D topological field theories can be extended to points? Specifically can quantum Chern-Simmons theory be extended to points?*

In this dissertation we study 2-dimensional extended topological field theories. We prove the following theorem:

Theorem 1.3.3 (Classification of Unoriented Topological Field Theories)**.** *The 2-dimensional unoriented bordism bicategory has the following presentation as a symmetric monoidal bicategory: It has the generators depicted in Figure 1.2 subject to the relations depicted in Figure 1.3.*

In dimension 2, the Folklore Theorem 1.1.1 implies that 2-dimensional topological field theories do indeed distinguish 2-manifolds. This theorem says that these field theories

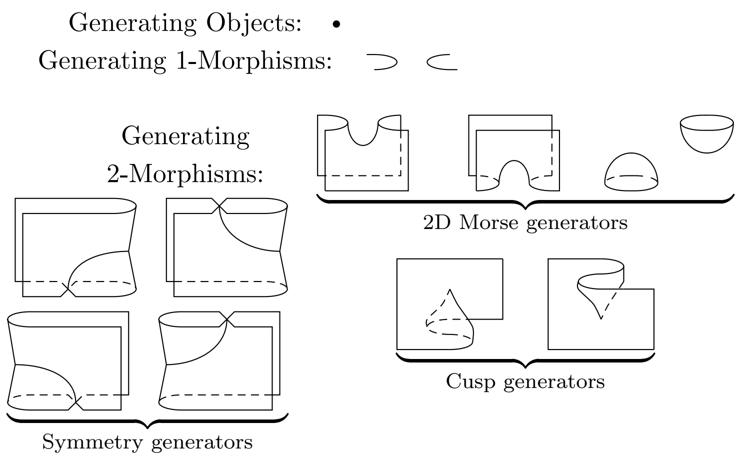


Figure 1.2: Generating Objects

are the same as commutative Frobenius algebras, which are well understood and plentiful. By writing down a few examples of Frobenius algebra it is not difficult to distinguish all surfaces. In order to answer the above open questions, or to have applications to *non-topological* field theories, it would highly desirable to have a proof of Theorem 1.1.1 and Theorem 1.3.3 which generalizes to higher dimensions. Unfortunately the proof of Theorem 1.1.1 which is usually given has essentially no hope of generalizing beyond 2-dimensions.

All published proofs of Theorem 1.1.1 proceed in the following way. First one uses Morse theory to deduce that any surface bordism can be decomposed into a composition of (disjoint unions of) the elementary bordisms given in Figure 1.1. Hence those elementary bordisms generate the bordism category. The difficulty is then proving that the obvious relations are in fact sufficient, i.e. that two surface built from the elementary bordisms are diffeomorphic if and only if they can be related by a finite number of these relations.

The usual proof of this, as suggested by L. Abrams [Abr96] and fully explored by J. Kock [Koc04], is to appeal to the classification of surfaces. The classification of oriented surfaces says that two closed oriented surfaces are diffeomorphic if and only if they have the same genus. There is an analogous statement for surfaces with boundary. This allows one to introduce a *normal form* for each surface, and one merely needs to prove that the relations are sufficient to reduce to the normal form. This can be done in an ad hoc fashion.

The problem with this method of proof is that it relies essentially on the classification of surfaces, and hence has no hope of generalizing to higher dimensions. While it is

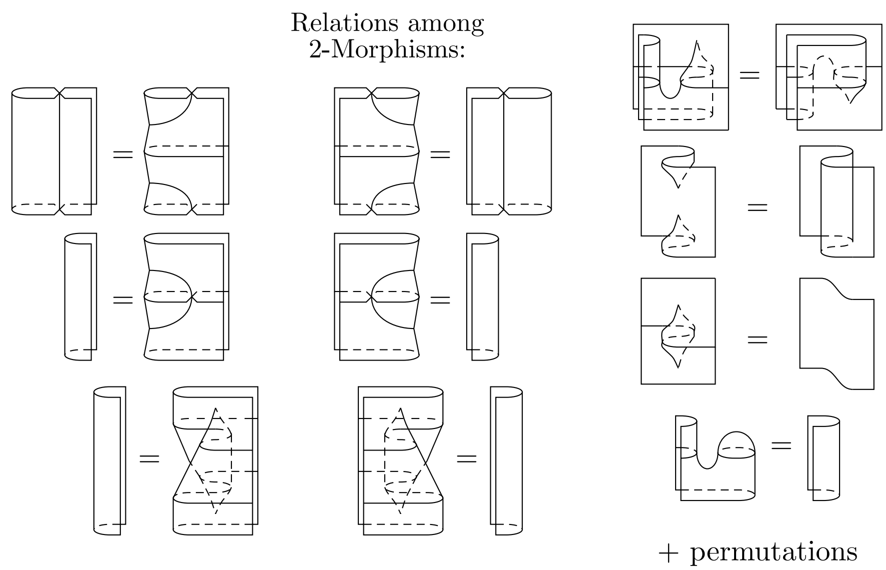


Figure 1.3: Relations

possible to give a proof of main theorems of this dissertation along these lines, such a proof would have little value.

Fortunately, there is a second proof of the Folk Theorem 1.1.1, which is the that is generalized here. As before it begins by looking at Morse functions for surfaces to derive generators. A Morse function (which is a generic map to $\mathbb{R}$) gives a linear, and hence 1-categorical way to decompose a surface. Analyzing the singularities of Morse functions gives 1-categorical generators. Each Morse function gives a decomposition into elementary generators, and conversely such a decomposition corresponds to a Morse function. Thus determining a sufficient set of relations to pass from one decomposition to another is tantamount to determining a sufficient set of relations relating any two Morse functions.

The new insight is to not use the classification of surfaces to get sufficient relations, but to use Cerf theory [Cer70]. The part of Cerf theory that is relevant is essentially a *family* version of Morse theory (a.k.a. parametrized Morse theory). In particular, a path of functions is a 1-parameter family of functions. Cerf theory then studies generic *paths* of functions. A generic path of functions consists of Morse functions for all but a finite number of isolated times. Analyzing these critical times more closely, we see that any two Morse functions can be related by a finite number of handle slides and birth/death moves.

This proves the sufficiency of the relations. This proof approach was suggested by S. Sawin [Saw95], but full details appear in the work of Moore and Segal, in the appendix of [MS06].

With the higher categorical machinery described in the last section in place, we may reduce the proof of Theorem 1.3.3 to a similar geometric 2-categorical generators and relations problem. The main theorems of Chapter 3 give a precise meaning to the abstract free symmetric monoidal bicategory with generators as in Figure 1.2 subject to the relations depicted in Figure 1.3. By drawing these abstract generators as particular bordisms, we are specifying their image in the bordism bicategory under a particular map. The universal property of the abstract symmetric monoidal bicategory ensures that there is a symmetric monoidal homomorphism from it to the bordism bicategory. Whitehead's theorem for symmetric monoidal bicategories then gives a list of three criteria for this symmetric monoidal homomorphism to be an equivalence. The content of Theorem 1.3.3 is that these criteria are satisfied.

The first two criteria are easily checked. The real heart of the argument is to show that the above functor is fully faithful on 2-morphisms. Fullness is equivalent to the statement that any 2-dimensional bordism can be decomposed in a (symmetric monoidal) 2-categorical fashion into a composition of the elementary bordisms of Figure 1.2. Faithfulness is equivalent to the statement that the relations of Figure 1.3 are sufficient to pass between any two such decompositions. We prove these statements by a geometric argument generalizing the techniques and results of Cerf theory. We have gone to some effort to split the argument into those portions which are purely topological and are independent of the framework of symmetric monoidal bicategories (Chapter 2) and those which are not (Chapter 4).

The relevant theorems of Cerf theory can be proven using classical jet transversality methods developed by Thom, Boardman, Mather, and others. These techniques can be adapted and generalized to obtain a 2-categorical generators and relations decomposition theorem as well. This is the subject of Chapter 2. More specifically, given manifolds X and Y, one may consider the *jet bundles* for maps from X to Y.

$$\begin{tikzcd} J^k(X,Y) \arrow[d] \\ X \arrow[u, dashed, bend left, "j^k f"] \end{tikzcd}$$

A given smooth map, $f : X \to Y$, it induces a section, $j^k f$, of this bundle. If we stratify the

jet bundle in some way, then Thom's jet transversality theorem ensures that the for generic set of maps the jet sections, $j^k f$, are transverse to these strata. These strata effectively classify the singularity type of the map f. If the strata are chosen judiciously, then one may derive normal coordinates around any given singularity. A more careful analysis allows one to piece together this local data to determine a global decomposition of X.

For Morse theory and Cerf theory one studies the jet bundles $J^k(X, \mathbb{R})$ and $J^k(X \times I, \mathbb{R} \times I)$, respectively. The latter is important because a path of functions can be regarded as a particular kind of map $X \times I \to \mathbb{R} \times I$. The Thom-Boardman stratification of the jet bundles is essentially sufficient to derive the basic results of Cerf theory, in this case.

In the 2-categorical setting, we want to have a 2-dimension, 2-categorical decomposition of our surface, Σ. For this reason we study the jets bundles $J^k(\Sigma, \mathbb{R}^2)$. The usual classification of singularities, using the Thom-Boardman stratification, is not quite sufficient to obtain a 2-categorical decomposition. Instead we use the projection $\mathbb{R}^2 \to \mathbb{R}$ to obtain a *finer stratification* of the jet bundles, and hence a finer classification of singularities. This new stratification and the normal coordinates for each singularity type are discussed in Chapter 2 and yield a complete list of generators for the bordism bicategory.

To obtain relations, we study paths of maps to $\mathbb{R}^2$. More precisely we study maps $f : \Sigma \times I \to \mathbb{R}^2 \times I$. We use the projections $\mathbb{R}^2 \times I \to \mathbb{R} \times I \to I$ to stratify the jet bundles $J^r(\Sigma \times I, \mathbb{R}^2 \times I)$ and thereby classify the generic singularities of maps f. Any two generic maps $\Sigma \to \mathbb{R}^2$ may be connected by a generic map $f : \Sigma \times I \to \mathbb{R}^2 \times I$, and the analysis of the singularities of f give us a finite list of local moves which allow us to pas from one 2-categorical decomposition to another. This gives us the sufficiency of the relations. These results are summarized in the main planar decomposition theorems of Chapter 2, Theorems 2.5.6 and 2.5.7.

The results of Chapter 2, while intended for bicategorical applications, are independent of the results on symmetric monoidal bicategories discussed in Chapter 3 (indeed they come first in the organization of this dissertation). In Chapter 4 we discuss results which unite our algebraic results on symmetric monoidal bicategories and the topological results of Chapter 2. First we introduce the symmetric monoidal bicategory of bordisms.

Previous attempts at defining the bordism bicategory have been made in [KL01] and [Mor07], but neither of these is quite adequate. First, they essentially ignore the *symmetric monoidal* structure on the bordism bicategory, which is essential to define extended topological field theories. This is perhaps understandable, since they did not have available

the results of Chapter 3. More importantly, neither [KL01] nor [Mor07], discuss the issue of gluing bordisms in sufficient detail to adapt their definition to a bicategory of bordisms with structure. The main issue is that when one glues two smooth manifolds along a common boundary, the resulting space, $W \cup_Y W'$, which is a topological manifold, does not have a unique smooth structure extending those on W and W'. While any two choices of such smooth structure result in diffeomorphic manifolds, this diffeomorphism is not *canonical*. In the 1-categorical bordism category, this issue is easily swept under the rug. But in the higher categorical setting it must be dealt with. Kerler-Lyubashenko [KL01] and Morton [Mor07], narrowly avoid this issue.

After introducing a naive version of the bordism bicategory, we give an improved version using manifolds which are equipped with a germ of a d-dimensional manifold containing them. These can be used for one method of solving the gluing problem and also for defining the bordism bicategory with structure. The type of structures that can be used are very general, and includes orientations, spin structures, and G-principal bundles.

Next we prove the main generators and relations classification theorem in the unoriented case, and show how to adapt the techniques of this dissertation to obtain generators and relations for the bordism bicategory equipped with structures. We use the oriented case as a particular example. Finally we discuss some general consequences and application of these classification results. Topological quantum field theories typically have linear or algebraic target categories, and in the extended context a good example of such a target bicategory is given by the bicategory Alg^2. The objects are algebras, the 1-morphisms from A to B are A-B-bimodules with composition given by tensor product of bimodules, and the 2-morphisms are given by bimodule maps. We use the generators and relations results obtained earlier in this chapter to completely classify extended topological field theories with this target, in both the oriented and unoriented settings. We identify the results with familiar classical notions in algebra.

Finally, we would be remiss if we did not discuss future applications of this work and current related research. As we have emphasized throughout. The importance of this work is not in the 2-dimensional classification result itself, but in the developed techniques and supporting lemmas. While the 2-dimensional case is interesting, the higher dimensional cases are even more so. There are a few variations that are easy to imagine, one may study d-dimensional bicategorical theories in which $d > 2$. The Reshetikhin-Turaev construction produces such a theory. One may also d-dimensional theories which are d-categorical, hence

entirely local and extended all the way to points. Again dimension 3 appears to be the most immediately interesting. The techniques developed here are currently being adapted in both these directions in joint work, [DSP].

As for related results, there is the recent exciting work of M. Hopkins and J. Lurie, which classifies a large class of fully extended topological field theories with sturcture, in all dimensions. Full details of this work have not been made public, but an expository article was released by Lurie in Jan. 2007 [Lur09]. When published, these result will imply the oriented and unoriented classification results given here. Hopkins and Lurie study a more elaborate version of the extended bordism n-category. Rather then taking the top dimensional bordisms up to diffeomorphism, they incorporates the homotopy type of the diffeomorphism groups through the language of (∞, n)-categories. These are a particular kind of ∞-category. Their proof, which is quite different from ours, involves a delicate induction which plays several variations of the bordism category off each other (in particular the framed bordism category and the unoriented bordism category). Moreover, their proof makes use of sophisticated homotopy theoretic techniques, rather then the more elementary approach outline here, and the statement of their results is expressed in homotopy theoretic language.

These two approaches to the classification theorem are well matched. While the Hopkins-Lurie approach yields stronger and more conceptual results, our results are more concrete and in many cases are better adapted for constructing examples. Moreover, the techniques of the Hopkins-Lurie approach seem to require that the extended topological field theories are *fully extended* in the sense that they go all the way down to points. There is no such restriction for the techniques we develop. These results compliment each other, and we hope both will be important in the exciting future of extended quantum field theory.

Chapter 2

Planar Decompositions of 2-Manifolds

In this chapter we prove the main geometric theorems used in the classification of 2-dimensional topological field theories. We formulate these results completely in the classical language of differential topology, and in particular make no mention of 2-dimensional bordism bicategory itself. In particular the results of this chapter are independent of the specifics of the construction of the 2-dimensional bordism bicategory, a task which is taken up in Chapter 4. Jet transversality, Cerf theory, and singularity theory have had a relatively long and fruitful mathematical life, and the results of this chapter are heavily influenced by analogous results in these fields. The techniques used here are essentially classical, building on well known and heavily developed existing results.

With this in mind, we begin with a review of the classical literature. In Sections 2.1.1 and 2.1.2 we review the classical notion of jet bundles and R. Thom's jet transversality theorems, including the Parametric Transversality Lemma 2.1.4. These results are completely standard and may be found, for example, in the text [GG73].

The main results of this chapter closely parallel standard results in Cerf theory, in particular the Cerf theory of 1-manifolds. We exploit this analogy, and in Sections 2.2.1 and 2.2.2 we provide a revisionist review of Morse theory and Cerf theory. This allows us to explain the main results, as well as introduce the key methods of proving them. For simplicity, we restrict our attention to the case of Morse and Cerf theory of 1-manifolds, which is the most analogous to the results obtained later in this chapter. While these

Morse and Cerf theory results are completely classical, we don't know of a reference which presents this material from precisely this viewpoint while simultaneously providing details at the necessary level. However the discussion of singularity theory given in the text [GG73] goes most of the way there. The proof techniques are essentially taken from there.

At a very informal level, Morse theory is the study of generic functions from a manifold M to $\mathbb{R}$. Cerf theory is then the study of generic *families* of functions from M to $\mathbb{R}$. One such family is given by a path of functions. One of the main results of Cerf theory is that a generic path of functions is in fact a path of Morse functions for all but a finite number of isolated critical times. Moreover at these critical times, there exist standard normal coordinates, in analogy with the results of the Morse Lemma. One may then use these results to study precisely how to change any given Morse function on M into any other Morse function. One of the main techniques is to regard a path of functions as a single map $M \times I \to \mathbb{R} \times I$. In this way, generic paths of functions are related to generic maps into $\mathbb{R} \times I$ or $\mathbb{R}^2$, and hence to the singularity theory of such maps.

As we will explain more precisely in the sections below, a map from M to $\mathbb{R}$ gives a 1-dimensional/1-categorical decomposition of M. The goal of this chapter is to provide a categorification of this idea. We want a 2-categorical/2-dimensional decomposition of M, and so we are led to study maps from M to $\mathbb{R}^2$, and generic paths of such maps. If our goal were merely to study generic maps to $\mathbb{R}^2$ and generic paths of such maps, then we could appeal to well established results in singularity theory. Indeed the classification of singularities in these dimensions is a well understood, by now classical subject. However, we intend to use these results to obtain new 2-categorical information. It is not enough merely to study the singularities of maps $M \to \mathbb{R}^2$, but we must simultaneously study the singularities of the composite map $M \to \mathbb{R}^2 \to \mathbb{R}$, where the last map is a standard projection. The singularities and ensuing decomposition theorems can then, in Chapter 4, be matched with the bicategorical framework.

The proofs of the key results of Cerf theory that we are mimicking proceed by stratifying the jet space by a suitable collection of submanifolds, for example by the Thom-Boardman stratification, [GG73, §5]. Then one proves a theorem analogous to the Morse Lemma, providing normal coordinates for maps whose jet sections, $j^k f$, are transverse to this stratification, see Section 2.2.2. Because we are simultaneously considering the singularities of maps $M \to \mathbb{R}^2$ and the projection $M \to \mathbb{R}^2 \to \mathbb{R}$, we are lead to a refinement of the usual Thom-Boardman stratification, and consequently to a finer classification of

the possible singularities. In Sections 2.2.3, 2.2.4, 2.2.5, 2.2.6, and 2.2.7 we introduce the precise stratification of the jet spaces that we will use, and derive the corresponding normal coordinates. This results in a finer classification of singularities than the standard classification.

In Section 2.3, we examine more closely the geometric structure of each of these singularities. Borrowing terminology from Cerf theory, we introduce the concept of a *graphic*. The graphic of a generic map is the image of the singular loci in the target, $\mathbb{R}^2$ or $\mathbb{R}^2 \times I$. These images consist of a collection of immersed surfaces, arcs, and points. In Section 2.4, we use multi-jet techniques to introduce additional strata. A map whose mulit-jet section is additionally tranverse to these strata, which holds for a dense collection of maps, has the property that these immersed surface, arcs, and points are in general position. We use this as the basis for defining an abstract graphic as a suitably labeled collection of surfaces, arcs, and points immersed in $\mathbb{R}^2$ or $\mathbb{R}^2 \times I$, and in general position.

Finally, in Section 2.5 we prove the main theorems of this section. We introduce the notion of planar diagrams and 3D diagrams. A planar diagram is roughly an abstract graphic, equipped with some additional data ("sheet data"). A generic map from a surface Σ to $\mathbb{R}^2$, with a few additional choices, induces a planar diagram. The additional data of a planar diagram is essentially combinatorial in nature, nevertheless given a planar diagram one can in fact recover a surfaces with a generic map to $\mathbb{R}^2$ inducing the original planar diagram. Moreover this surface is unique up to isomorphism. A similar analysis holds for generic maps $\Sigma \times I \to \mathbb{R}^2 \times I$, and this allows us to introduce a equivalence relation an planar diagrams. Two planar diagrams are equivalent if they can be related by a finite number of "local relations", and in fact surface up to isomorphism are in bijection with equivalence classes of planar diagrams. These results are encapsulated in Theorem 2.5.7.

2.1 Jet Bundles and Jet Transversality

Jets and jet bundles were introduced in 1951 by Ehresmann [Ehr51a, Ehr51b, Ehr51c] and over the nearly 60 years since their inception have become an increasingly important tool in differential geometry and the study of PDEs. In singularity theory they have a particularly prominent role, and, when combined with Thom's jet transversality theorem, have led to a classification of singularities of small codimension.

Roughly speaking, two functions $f, g : X \to Y$ have the same r-jet at a point x

if they agree "up to order r". There are several equivalent ways to make this precise. A common method is to choose local coordinates around x and $f(x) = y = g(x)$. Then f and g agree up to order r (at x) if all of their partial derivatives up to order r agree. For example, f and g agree up to 1^{st} order at x if $f(x) = g(x)$ and the matrix of single partial derivatives of f and g agree at x. This is precisely the condition

$$df_x = dg_x.$$

This condition is coordinate independent, and similarly the condition that f and g agree up to order r is also coordinate independent, which is not immediate from the above description.

There is an alternative approach to jets and jet bundles which is algebraic. This approach has the advantage that it is manifestly functorial and coordinate free. However, in the local coordinate approach it is straightforward to show that the jets form a smooth manifold and a bundle over $X \times Y$. For a particular chart this is clear: the jets consist of the (trivial) bundle of possible values of the partial derivatives. Then one must show how these glue together when a change of coordinates is preformed. This is more difficult to show in the algebraic approach. Both approaches have their respective advantages and disadvantages and we will employ both points of view in what follows. The interested reader should consult [GG73, KMS93].

Continuing the example of 1-jets, we have that the one-jets of maps from X to Y form the bundle

$$J^1(X,Y) = p_1^*(T^*X) \otimes p_2^*(TY) \to X \times Y,$$

where p_i is projection onto the i^{th} factor. In this case the jets form a vector bundle over the space $X \times Y$, but in general this will not be the case. Via the projection $X \times Y \to X$, we can view the jet bundle as a bundle over X. Then any smooth function $f : X \to Y$ gives us a smooth section,

$$df : X \to J^1(X,Y)$$

whose value at x is precisely the differential df_x. This pattern persists. A map $f : X \to Y$ gives a section, $j^r f$, of the bundle of r-jets over X, $J^r(X,Y)$.

Classical transversality has several formulations, but one version concerns the density of certain maps between smooth manifolds. Given smooth manifolds, X and Y, and a smooth submanifold $W \subset Y$, the transversality theorems assert that the maps $f : X \to Y$ which are transverse to W are dense in the space of all maps, and that for such a map, $f^{-1}(W)$ is a smooth submanifold of X of the expected codimension.

Thom transverality (a.k.a. jet transversality) is a statement about the sections, $j^r f$, induced by maps $f : X \to Y$. Given a submanifold $W \subset J^r(X, Y)$, the jet transversality theorem says that for a dense subset of maps $f : X \to Y$, the sections $j^r f : X \to J^r(X, Y)$ are transverse to W.

2.1.1 Jet Bundles

Let X and Y be smooth manifolds. Given a point $x \in X$, we can consider the algebra $C^\infty_x(X)$ of germs of functions at x. This is a local ring with unique maximal ideal $\mathfrak{m}_x$ consisting of the germs of functions which vanish at x. A smooth map $X \to Y$ induces a map of local algebras $C^\infty(Y) \to C^\infty(X)$, and consequently for each integer $k \geq 0$ we have an induced map of rings

$$f^*_x : C^\infty_y/\mathfrak{m}^{k+1}_y \to C^\infty_x/\mathfrak{m}^{k+1}_x.$$

Moreover, f need not be globally defined. If $U \subset X$ is a neighborhood of x, and $V \subset Y$ is a neighborhood of y, then any smooth map $f : U \to V$, mapping x to y induces a map of rings, f^*_x, as above. The map f^*_x only depends on the germ of f near x. This construction is also functorial. If f is a germ of a map sending $x \in X$ to $y \in Y$ and g is a germ of a map sending $y \in Y$ to $z \in Z$, then the composition

$$C^\infty_z/\mathfrak{m}^{k+1}_z \xrightarrow{g^*_x} C^\infty_y/\mathfrak{m}^{k+1}_y \xrightarrow{f^*_x} C^\infty_x/\mathfrak{m}^{k+1}_x$$

coincides with the homomorphism $(g \circ f)^*_x$.

Fix an integer $r \geq 0$. We impose an equivalence relation on triples (x, U, f), where $x \in X$, $U \subset X$ is a neighborhood of x and $f : U \to Y$ is a smooth map. We set $(x, U, f) \sim_r (x', U', f')$ if $x = x'$, $f(x) = f'(x)$ (whose common value we denote y), and $f^*_x = f'^*_x \in \mathrm{Hom}(C^\infty_y/\mathfrak{m}^{r+1}_y, C^\infty_x/\mathfrak{m}^{r+1}_x)$. If this is the case, we say that f agrees with f' at x up to r^{th} order. An equivalence class is called a *jet* at x, and the set of equivalence classes is denoted $J^r(X, Y)$. For the equivalence relation we could just as well have replaced the pair (U, f) with the germ of f at x. However by not using germs we see that there are well defined quotient maps,

$$U \times C^\infty(U, Y) \to J^r(X, Y).$$

These quotient maps are compatible in the sense that whenever $V \subset U \subset X$, the following diagram commutes:

$$\begin{tikzcd} & V \times C^\infty(U,Y) \arrow[dl] \arrow[dr] & \\ V \times C^\infty(V,Y) \arrow[dr, "q"'] & & U \times C^\infty(U,Y) \arrow[dl, "q"] \\ & J^r(X,Y) & \end{tikzcd}$$

More generally for a smooth manifold, S, we may consider S-families of smooth maps $U \to Y$, by which we mean smooth maps $S \times U \to Y$. We have a sequence of maps of sets,

$$S \times U \times C^\infty(S \times U, Y) \xrightarrow{\text{ev}} U \times C^\infty(U,Y) \xrightarrow{q} J^r(X,Y),$$

which we may use to define a topology on the spaces $J^r(X,Y)$. By adjunction this gives a map of sets,

$$C^\infty(S \times U, Y) \to Map(S \times U, J^r(X,Y))$$

where $Map(S \times U, J^r(X,Y))$ denotes the set theoretic maps. Denote the image by $W_{S,U}$. We give $J^r(X,Y)$ the finest topology in which each of the maps in $W_{S,U}$ is continuous for each manifold S and for each open $U \subset X$.

In particular, this implies the there is a well defined, continuous map $J^k(X,Y) \to X$, which we call the projection to X. Moreover, taking the case $U = X$, if $f : X \to Y$ is a smooth function, we get a continuous map,

$$X = X \times \{f\} \hookrightarrow X \times C^\infty(X,Y) \to J^k(X,Y).$$

We denote this map by $j^k f : X \to J^k(X,Y)$. By construction it is a section of the projection map.

It is a classical result [GG73, KMS93] that the jet space, as defined above, agrees as a topological space with the jet bundle constructed using local coordinates. Thus the jet space is in fact a finite dimensional smooth manifold and a fiber bundle over the product $X \times Y$. In fact, associated to the pair X and Y is a tower of smooth fiber bundles

$$X \leftarrow X \times Y = J^0(X,Y) \leftarrow J^1(X,Y) \leftarrow J^2(X,Y) \leftarrow J^3(X,Y) \leftarrow \cdots$$

which we call the *jet bundles*. In the local coordinates picture these successive maps can be viewed as the process of forgetting the highest order partial derivatives. In the algebraic

picture it is the process of taking the quotient by a larger ideal. With this smooth structure, $j^k f$ becomes a smooth section of the projection map.

If we fix a point $x \in X$, we can consider the fiber ${}_xJ^r(X,Y)$, i.e those jets whose projection to X is the point x. Similarly, we can consider $J^r(X,Y)_y$ consisting of all jets whose projection to Y has value y. We can also consider the fiber over $(x,y) \in X \times Y$, denoted ${}_xJ^r(X,Y)_y = {}_xJ^r(X,Y) \cap J^r(X,Y)_y$. For a fixed $y \in Y$, the spaces $J^r(X,Y)_y$ form bundles over X and the spaces ${}_xJ^r(X,Y)$ form bundles over Y, for fixed x.

Let us return once again to the case of 1-jets. We argued from the coordinate theoretic approach that the 1-jets can be identified with the vector bundle

$$J^1(X,Y) = p_1^*(T^*X) \otimes p_2^*(TY) \to X \times Y.$$

This can also be seen from the algebraic approach as well. The inclusion of the (germs of) constant functions into C_x^∞ allows us to canonically split the exact sequence,

$$\mathfrak{m}_x/\mathfrak{m}_x^2 \to C_x^\infty/\mathfrak{m}_x^2 \to \mathbb{R} = C_x^\infty/\mathfrak{m}_x.$$

If we have a vector space V, then $\mathbb{R} \oplus V$ becomes an algebra via the multiplication $(t_0, v_0) \cdot (t_1, v_1) = (t_0t_1, t_0v_1 + t_1v_0)$ and $C_x^\infty/\mathfrak{m}_x^2$ is isomorphic to such an algebra. In fact, the above splitting gives us a canonical algebra isomorphism $C_x^\infty/\mathfrak{m}_x^2 \cong \mathbb{R} \oplus (\mathfrak{m}_x/\mathfrak{m}_x^2) \cong \mathbb{R} \oplus T_x^*X$. An easy calculation shows that algebra homomorphisms $\mathbb{R} \oplus V \to \mathbb{R} \oplus W$ are in bijection with linear maps $V \to W$. Thus we find that the fibers ${}_xJ^1(X,Y)_y$, calculated from the algebriac perspective, are the algebra homomorphisms

$$\mathbb{R} \oplus T_y^*Y \to \mathbb{R} \oplus T_x^*X,$$

and so consist of precisely the space $T_x^*X \otimes T_yY$, as expected.

One of the advantages of introducing the jet bundles is that they provide a means of combining both geometric and algebraic understanding. There are two common ways of defining the tangent bundle to a space: either as equivalence classes of certain curves $\gamma : \mathbb{R} \to Y$ or as the derivations of the algebra of germs C_y^∞. Both of these perspectives are manifest from the point of view of jet bundles. The above discussion of 1-jets implies that we have the following isomorphism

$${}_0J^1(\mathbb{R},Y)_y \cong T_0^*\mathbb{R} \otimes T_yY \cong T_yY.$$

Moreover the left hand side can be viewed in two ways, corresponding to the two ways of defining the tangent bundle. On the one hand it is defined as an equivalence class of maps

$\gamma : \mathbb{R} \to Y$ (i.e. equivalence classes of curves); it is precisely those curves sending 0 to y, where two are considered equivalent if they agree at y up to 1st order. But as we have seen, this fiber is also the algebra homomorphisms,

$$C_y^\infty/\mathfrak{m}_y^2 \to C_0^\infty/\mathfrak{m}_0^2 \cong \mathbb{R}[x]/(x^2).$$

Such a homomorphism induces the composite homomorphism $C_y^\infty \to \mathbb{R}[x]/(x^2)$. Let us consider these more closely. Such a homomorphism is given by the formula,

$$h : f \mapsto f(y) + \xi_h(f) \cdot x.$$

The fact that this is an algebra homomorphism implies that

$$\begin{aligned}
(fg)(y) + \xi_h(fg) \cdot x &= h(fg) \\
&= h(f)h(g) \\
&= (f(y) + \xi_h(f) \cdot x) \cdot (g(y) + \xi_h(g) \cdot x) \\
&= f(y)g(y) + (\xi_h(f)g(y) + f(y)\xi_h(g)) \cdot x.
\end{aligned}$$

Thus we find that h is equivalent to the algebra derivation ξ_h. If $f \in \mathfrak{m}_y$, (so that $f(y) = 0$), then we find that

$$h(f^2) = f^2(y) + 2f(y)\xi_h(f) \cdot x = 0.$$

Thus any algebra homomorphism h factors as

$$h : C_y^\infty \to C_y^\infty/\mathfrak{m}_y^2 \to \mathbb{R}[x]/(x^2).$$

This identifies the tangent bundle, i.e. the fiber ${}_0J^1(\mathbb{R}, Y)_y$, with the algebra derivations of C_y^∞, precisely the second definition of the tangent space. A similar discussion shows that the cotangent bundle is $J^1(X, \mathbb{R})_0 \to X$.

The higher order jet spaces ${}_0J^r(\mathbb{R}, Y)$ and $J^r(X, \mathbb{R})_0$ should be thought of as higher order analogs of the tangent and cotangent bundles. The space $\mathbb{R}$ is a topological ring and so, by the functoriality of the jet bundles, the fibers ${}_xJ^r(X, \mathbb{R})_0$ become rings, in fact algebras over $\mathbb{R}$. These algebras coincide with the algebras $C_x^\infty/\mathfrak{m}_x^{r+1}$, as can be readily checked, see [KMS93] as well.

There are several closely related bundles, such as $J^k(X^n, \mathbb{R}^n)_0$ and ${}_0J^k(\mathbb{R}^n, Y^n)$ but these are better understood using the frame bundle of order r. It is well known that one can associate a GL_n-principle bundle to the tangent bundle (or to the cotangent bundle) of

a manifold Y of dimension n. This is given by taking the bundle of n-frames whose fibers at a point y consist of the sets of n-linearly independent vectors in T_yY. This fiber consists exactly of the linear isomorphisms

$$\mathbb{R}^n \cong T_0\mathbb{R}^n \to T_yY.$$

Equivalently these are the jets in ${}_0J^1(\mathbb{R}^n, Y)_y$ which are represented by a *local diffeomorphism.*

We define the *k^{th}-order frame bundle* P^kY as the sub-bundle of ${}_0J^k(\mathbb{R}^n, Y^n)$ whose fiber consists of those k-jets represented by *local diffeomorphisms.* This is a G^k_n-principal bundle over Y, where $G^k_n \subset {}_0J^k(\mathbb{R}^n, \mathbb{R}^n)_0$ is the *jet group* of k^{th}-order jets represented by local diffeomorphisms. Let us recall some facts about the jet groups. We have the isomorphism $GL_m = G^1_m$, familiar from the frame bundle construction. In general the projection ${}_0J^k(\mathbb{R}^n, \mathbb{R}^n)_0 \to {}_0J^1(\mathbb{R}^n, \mathbb{R}^n)_0$ gives us a (split) exact sequence of Lie groups,

$$1 \to B^k_m \to G^k_m \to GL_m \to 1$$

The normal subgroup B^k_m is connected, simply connected, and nilpotent. A more detailed discussion can be found in [KMS93].

The vector space structure on $\mathbb{R}^n$ induces a vector space structure on $L^k_{m,n} = {}_0J^k(\mathbb{R}^m, \mathbb{R}^n)_0$ and G^k_n naturally "acts"on the right on this space. This action is generally *not* a linear action, but we can still form the associated bundle. Given a manifold Y, we can take the k^{th}-order frame bundle of Y, which is a (left) G^k_n-principal bundle and form the bundle associated to the "representation"$L^k_{m,n}$. This is precisely the bundle ${}_0J^k(\mathbb{R}^m, Y)$.

A similar discussion applies to coframes. The *k^{th}-order coframe bundle* $P^{k*}X$ of X^m consists of those jets in $J^k(X^m, \mathbb{R}^m)_0$ which are represented by local diffeomorphisms. This is a right G^k_m-principal bundle. $L^k_{m,n}$ has a natural left G^k_m-action and the corresponding associated bundle is $J^k(X, \mathbb{R}^n)_0$. Moreover the left action of G^k_m and the right action of G^k_n on $L^k_{m,n}$ are compatible. We can view this as a single right action by $G^k_m \times G^k_n$ in the usual way. The bundle $P^kX \times P^k(Y) \to X \times Y$ is a principal $G^k_m \times G^k_n$-bundle and hence we can form the bundle associated to $L^k_{m,n}$. This is the bundle $J^k(X^m, Y^n)$ over $X \times Y$, whose typical fiber $L^k_{m,n}$ has dimension,

$$n\left[\begin{pmatrix} m+k \\ m \end{pmatrix} - 1\right].$$

In fact $L^k_{m,n}$ can be canonically identified with the space of n-tuples of polynomials in m-variables of degree $\leq k$, whose constant coefficient vanishes. Choosing sufficiently small coordinate patches U and V in both X and Y, we can trivialize the jet bundle over $U \times V$. It can be identified with the trivial bundle $U \times V \times L^k_{m,n}$. These trivializations are then glued together by the action of the jet group.

In this work we will also need to consider manifolds with boundary and corners. Jets and jet bundles for manifolds with corners are slightly more delicate, but are also well established [Mic80]. We will only need to consider the case when the source manifold, X, has corners but the target manifold, Y, does not. Moreover, we will only consider those manifolds with corners, X^m, which are equipped with a germ in a neighborhood of $X \subset \tilde{X}$. Here $\tilde{X}$ is an m-manifold without boundary, and X is a submanifold with corners of $\tilde{X}$. In this case we may define the jet bundle $J^k(X, Y)$ as the pullback to X of the bundle $J^k(\tilde{X}, Y)$ over $\tilde{X}$.

Multi-jet bundles will also play and important role, and are defined similarly. Let X be a smooth manifold. Define $X^{(s)} \subset X^s$ to be $X^{(s)} = \{(x_1, \ldots, x_s) \in X^s \mid x_i \neq x_j \text{ for } 1 \leq i < j \leq s\}$. Let $\mathbf{k} = (k_1, \ldots, k_s)$ be a multi-index of natural numbers and let $J^{\mathbf{k}}(X, Y) = J^{k_1}(X, Y) \times \cdots \times J^{k_s}(X, Y)$. Let $p : J^k(X, Y) \to X$ be the projection map and define $p^{\mathbf{k}} : J^{\mathbf{k}}(X, Y) \to X^s$ to be the obvious projection map. Then the s-fold $\mathbf{k}$-jet bundle is defined to be $J^{(\mathbf{k})}(X, Y) = (p^{\mathbf{k}})^{-1}(X^{(s)})$. It is a smooth bundle over the smooth manifold $X^{(s)}$. Now let $f : X \to Y$ be a smooth map. Then we define $j^{\mathbf{k}} f : X^{(s)} \to J^{(\mathbf{k})}(X, Y)$ to be the section,

$$j^{\mathbf{k}} f(x_1, \ldots, x_s) = (j^{k_1} f(x_1), \ldots, j^{k_s} f(x_s)).$$

2.1.2 Jet Transversality

The jet transversality theorems presented below are central for the results of this chapter. The essence of these theorems is that given any submanifold $W \subset J^k(X, Y)$, for a generic map $f : X \to Y$, the section $j^k f : X \to J^k(X, Y)$ is transverse to W. Moreover, if we have a countable collection of submanifolds $W_i \subset J^k(X, Y)$, we may require $j^k f$ to be transverse to each W_i separately, which will also hold for generic maps. These simple observations can have dramatic consequences. By judiciously choosing the W_i, we may prove that generic maps can be completely described locally. We can show that certain singularities occur and that others do not. In essence we will be able to achieve a local

classification of maps between certain spaces. Most of the following definitions and theorems can be found the classic texts [GG73, Mic80].

Definition 2.1.1. Let X and Y be smooth manifolds and $f : X \to Y$ a smooth map. Let W be a submanifold of Y and x a point in X. Then f *intersects* W *transversally at* x if either

1. $f(x) \notin W$, or
2. $f(x) \in W$ and $T_{f(x)}Y = T_{f(x)}W + (df)_x(T_xX)$.

If $A \subset X$, then f *intersects* W *transversally on* A if f intersects W transversally at x for all $x \in A$, and finally f *intersects* W *transversally* if f intersects W transversally on X, in which case we denote this by $f \pitchfork W$. $\diamond$

One of the key features of transversality is the following classical result.

Theorem 2.1.2. *Let X and Y be smooth manifolds, and $W \subset Y$ a smooth submanifold. Let $f : X \to Y$ be a smooth map and assume that $f \pitchfork W$. Then $f^{-1}(W)$ is a submanifold of X and* $\operatorname{codim} f^{-1}(W) = \operatorname{codim}(W)$.

In order to speak about generic mappings, we must be able to talk about dense subsets of mappings, and in order to talk about dense subsets we must place a topology on the space of mappings. The standard topology on $C^\infty(X, Y)$ that we will use is the Whitney C^∞-topology. Let S be a subset of $J^k(X, Y)$. We define the following set:

$$M(S) :=\{f \in C^\infty(\mathsf{X}, \mathsf{Y}) \mid j^k f(X) \subset S\}$$

Note that $\cap M(S_\alpha) = M(\cap S_\alpha)$, for an arbitrary collection of sets S_α. The collection of $M(U)$ where U ranges over all the open subsets of $J^k(X, Y)$ forms a basis for a topology on $C^\infty(X, Y)$, which we call the Whitney C^k-topology. The topology generated by the union of these topologies for all k is the Whitney C^∞-topology.

A subset R of a topological space X is called *residual* if it is the intersection of a countable collection of dense open sets. If X is a Baire space, then every residual set is dense. With the Whitney C^∞-topology, $C^\infty(X, Y)$ is a Baire space, and moreover the map,

$$j^k : C^\infty(X, Y) \to C^\infty(X, J^k(X, Y))$$

is a continuous map.

Theorem 2.1.3. *Let X and Y be smooth manifolds with W a submanifold of Y. Let $K \subset W$ be an open subset whose closure in Y is contained in W, and let $T_K = \{f \in C^\infty(X,Y) \mid f \pitchfork W \text{ on } \overline{K}\}$. Then T_K is an open subset of $C^\infty(X,Y)$ (in the Whitney C^∞-topology).*

The essence of transversality is the following parametric transversality lemma. It says that if a B-family of maps from X to Y is transverse to W, then for generic values of the parameter, $b \in B$, the individual maps $X \to Y$ are transverse to W. From this all manner of other transversality theorems follow.

Lemma 2.1.4 (Parametric Transversality Lemma)**.** *Let X, B, and J be smooth manifolds, with $W \subseteq J$ a smooth submanifold. Let $j : B \to C^\infty(X,J)$ be a map of sets and assume that $\Phi : X \times B \to J$, defined by $\Phi(x,b) = j(b)(x)$, is smooth and $\Phi \pitchfork W$. Then the set, $\{b \in B \mid j(b) \pitchfork W\}$ is dense in B.*

This classical lemma essentially follows from Brown's Theorem (a corollary of Sard's Theorem), and details of the proof may be found in [GG73, II §4 Lemma 4.6]. This is the key lemma which makes the proof of the following theorem possible.

Theorem 2.1.5 (Thom Transversality Theorem)**.** *Let X and Y be smooth manifolds and $\{W_i\}$ a countable collection of submanifolds of $J^k(X,Y)$. Let*

$$T_W = \{f \in C^\infty(X,Y) \mid j^k f \pitchfork W_i,\ \forall i\}.$$

Then T_W is a residual subset of $C^\infty(X,Y)$ in the Whitney C^∞-topology.

Proof Outline. First we choose a countable covering $\{W_{i,r}\}$ of each of the submanifolds W_i, such that each of the $W_{i,r}$ is open in W_i, it's closure is contained in W_i, and for which there exist suitably nice coordinates for X and Y around the image of $\overline{W}_{i,r}$ under the projection,

$$J^k(X,Y) \to X \times Y.$$

Then we consider the sets,

$$T_{W_{i,r}} = \{f \in C^\infty(X,Y) \mid j^k f \pitchfork W_i \text{ on } \overline{W}_{i,r}\}.$$

The set T_W is the intersection of the sets $T_{W_{i,r}}$, hence it is enough to prove that each is open and dense.

If $T_{i,r} = \{g \in C^\infty(X, J^k(X,Y)) \mid g \pitchfork W_i \text{ on } \overline{W}_{i,r}\}$, then $T_{i,r}$ is open by Theorem 2.1.3. Since $j^k : C^\infty(X,Y) \to C^\infty(X, J^k(X,Y))$ is continuous we have that $T_{W_{i,r}} = (j^k)^{-1}(T_{i,r})$ is also open.

Density is more difficult to prove. Given any map $f : X \to Y$, we let $B = \mathbb{R}^n \times L^k_{m,n}$, the space of all polynomial mappings $\mathbb{R}^m \to \mathbb{R}^n$ of degree k (not necessarily preserving the origin). Using our judicious choice of local coordinates in X and Y, we construct a deformation g_b of f, parametrized by $b \in B$. We define $\Phi : X \times B \to J^k(X,Y)$ by

$$\Phi(x,b) = j^k g_b(x).$$

Unfortunately, we cannot immediately apply Lemma 2.1.4, because the map Φ may not be transverse to W. However, if we shrink B to a suitable neighborhood of the origin, then Φ becomes a local diffeomorphism and hence satisfies any transversality condition. □

The reason for expounding on the proof of the Thom Transversality Theorem is that it is now clear that essentially identical theorems hold in much more generality. For example, the jet transversality theorem in the case where X has corners readily follows from the theorem for its neighborhood, $\tilde{X}$. A relative version also holds.

Theorem 2.1.6 (Thom Transversality Theorem (with corners))**.** *Let X be a smooth manifold with corners, $X \subset \tilde{X}$ a smooth embedding into a smooth manifold without corners, Y a smooth manifold, and $\{W_i\}$ a countable collection of submanifolds (possibly with corners) of $J^k(X,Y)$. Let*

$$T_W = \{f \in C^\infty(X,Y) \mid j^k f \pitchfork W_i,\ \forall i\}.$$

Then T_W is a residual subset of $C^\infty(X,Y)$ in the Whitney C^∞-topology.

Theorem 2.1.7 (Relative Thom Transversality Theorem)**.** *Let X be a smooth manifold, possibly with corners, equipped with a germ of embedding $X \subset \tilde{X}$ into a smooth manifold without corners. Let Y be a smooth manifold and let and $\{W_i\}$ a countable collection of submanifolds (possibly with corners) of $J^k(X \times I, Y \times I)$. Let $f_0, f_1 : X \to Y$ be smooth maps such that there exists a map $F : X \times I \to Y \times I$ which satisfies:*

1. *There exist neighborhoods N_0 of $0 \in I$, and N_1 of $1 \in I$, such that F satisfies $f_t(x) = f_0(x)$ for $t \in N_0$ and $f_t(x) = f_1(x)$ for $t \in N_1$.*

2. *$F \pitchfork W_i$ on a neighborhood of $X \times \{0,1\}$.*

Note that if there exists such an F, then property (2) holds for any F satisfying property (1). Let $C^{\infty}_{rel}(X \times I, Y \times I)$ denote the subspace of those maps F satisfying property (1), above. Let

$$T_W = \{F \in C^{\infty}_{rel}(X \times I, Y \times I) \mid j^k f \pitchfork W_i,\ \forall i\}.$$

Then T_W is a residual subset of $C^{\infty}_{rel}(X \times I, Y \times I)$ in the Whitney C^{∞}-topology.

There is one final version of jet transversality we will need, which also generalizes to both the case of manifolds with corners and the relative case.

Theorem 2.1.8 (Multi-Jet Transversality Theorem). *Let $\mathbf{k} = (k_1, \ldots, k_s)$ be a multi-index of natural numbers. Let X and Y be smooth manifolds with $\{W_i\}$ a countable collection of submanifolds of $J^{(\mathbf{k})}(X, Y)$. Let*

$$T_W = \{f \in C^{\infty}(X, Y) \mid j^{\mathbf{k}} f \pitchfork W_i,\ \forall i\}.$$

Then T_W is a residual subset of $C^{\infty}(X, Y)$.

2.2 Stratifications of the Jet Space

In this section we will begin to use the results of the previous section to classify generic maps between certain manifolds X and Y. Roughly what we will do is to choose a stratification of the jet space

$$J^k(X, Y) = \cup S_i,$$

then, by the Thom Transversalty Theorem 2.1.5, we are assured that a dense collection of maps will be simultaneously transverse to each of the strata S_i. We call such maps *generic*. By choosing the strata appropriately, we will be able to derive normal coordinates for these generic mappings, which will ultimately lead to the main result of this chapter, a planar decomposition theorem for surfaces. The techniques we will employ are essentially imported from Cerf theory, and so it is natural that we include a brief review of those aspects we will later generalize.

2.2.1 Rudimentary Morse Theory

Morse theory is a cornerstone of differential topology, and it would be difficult to improve upon the many wonderful expositions which cover it. Nevertheless, it will be useful to highlight what the previous sections on jet bundles and transversality have to offer.

In Morse theory one studies maps from a manifold X to $\mathbb{R}$, and so naturally we may consider the jet space $J^1(X, \mathbb{R})$. We can stratify $J^1(X, \mathbb{R})$ into two strata S_0 and S_1 based on the *corank* of the differential df. Thus S_0 corresponds to those jets where df has rank one, and S_1 to those for which df vanishes. S_0 is open and dense in $J^1(X, \mathbb{R})$ and S_1 has codimension equal to the dimension of X.

By Theorem 2.1.5, we know that for a dense subset of maps $f : X \to \mathbb{R}$, $j^1 f$ is transverse to both S_0 and S_1. This implies that for such generic f, the differential df vanishes at only isolated points in X (the critical points). We can obtain more refined information by stratifying the higher jet bundles.

Let x be a point of X and $f : X \to Y$. Then $j^k f : X \to J^k(X, Y)$, and we can consider its differential at x, i.e. the map

$$(dj^k f)_x : T_x X \to T_{j^k f(x)} J^k(X, Y).$$

This map is determined by the $(k+1)$-jet of f in the following sense, (see [GG73]). Let $\psi = j^{k+1} f(x)$ be the $(k+1)$-jet of f at x. Let g be any other (germ of a) map such that $j^{k+1} g(x) = \psi$, then

$$(dj^k f)_x = (dj^k g)_x,$$

which may be checked in local coordinates. Hence we may reconstruct the kernel of this map simply by knowing the $(k+1)$-jet.

Returning to the study of maps from X to $\mathbb{R}$, let us look at the jet space $J^2(X, \mathbb{R})$. The inverse images of S_0 and S_1 under the bundle projection,

$$J^2(X, Y) \to J^1(X, Y)$$

induce a preliminary stratification of $J^2(X, Y)$ (we will call these strata $S_0^{(2)}$ and $S_1^{(2)}$, respectively). $S_0^{(2)}$ is a perfectly fine stratum, however we may refine $S_1^{(2)}$ to obtain more useful information.

Recall that the fibers ${}_x J^1(X, \mathbb{R})_y$ may be identified with $\mathrm{Hom}(T_x X, T_y \mathbb{R})$. Under this identification, $S_1 \cap {}_x J^1(X, \mathbb{R})_y = \{0\} \subseteq \mathrm{Hom}(T_x X, T_y \mathbb{R})$. Thus, the normal bundle νS_1 of S_1 in $J^1(X, \mathbb{R})$ is canonically isomorphic to $\mathrm{Hom}(TX, T\mathbb{R})$, where by TX and $T\mathbb{R}$ we mean their corresponding pullbacks to S_1. Thus by our previous discussion, we get a map

of fiber bundles over S_1,

$$\begin{aligned} S_1^{(2)} \to \mathrm{Hom}(TX, TJ^1(X,Y)) \to &\mathrm{Hom}(TX, \nu S_1) \\ &\cong \mathrm{Hom}(TX, \mathrm{Hom}(TX, T\mathbb{R})) \\ &\cong \mathrm{Hom}(TX \otimes TX, T\mathbb{R}) \end{aligned}$$

where all the relevant vector bundles denote their corresponding pullbacks to S_1. The composition of these maps associates to $\sigma \in S_1^{(2)}$ the Hessian of f at x, where f is any map germ such that $j^2 f(x) = \sigma$ (and this doesn't depend on the representative f). Thus the above composition factors as a map of fiber bundles over S_1,

$$S_1^{(2)} \to \mathrm{Hom}(TX \odot TX, T\mathbb{R})$$

where $\odot$ denotes the symmetric tensor product. An easy calculation shows that this map is surjective submersion. We obtain a finer stratification of $S_1^{(2)}$ by stratifying $\mathrm{Hom}(TX \odot TX, T\mathbb{R})$ and pulling it back by the above map.

Let V and W be vector spaces, and let $L^r(V,W)$ denote the subset of linear maps $V \to W$ which "drop rank by r". That is, if R denotes the maximal possible rank for any map $V \to W$, then $L^r(V,W)$ consists of those maps which have rank $R - r$. This construction is clearly natural, and hence applies to vector bundles as well. A straightforward computation shows that the pullback (of fiber bundles over S_1),

$$\begin{tikzcd} \mathrm{Hom}(TX \odot TX, T\mathbb{R})_r \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & L^r(TX, \mathrm{Hom}(TX, TR)) \arrow[d] \\ \mathrm{Hom}(TX \odot TX, T\mathbb{R}) \arrow[r] & \mathrm{Hom}(TX, \mathrm{Hom}(TX, T\mathbb{R})) \end{tikzcd}$$

is again a smooth fiber bundle. This gives us a stratification of $\mathrm{Hom}(TX \odot TX, T\mathbb{R})$ by the submanifolds $\mathrm{Hom}(TX \odot TX, T\mathbb{R})_r$, and hence a corresponding stratification of $S_1^{(2)}$ by submanifolds $S_{1,r}$. The codimension of $S_{1,0}$ (which we call the *non-degenerate stratum*) is the dimension of X. The other strata $S_{1,r}$ (which we call *degenerate*) have higher codimension. If we wish, we may further divide $S_{1,0}$ into components according to the signature of the non-degenerate Hessian.

Now Theorem 2.1.5 implies that for generic f, not only are the only critical points isolated points in X, but that each of these critical points is non-degenerate.[1] Such a

[1] Due to the relative dimension of X and the codimension of the degenerate strata, transversality of $j^2 f$ and $S_{1,r}$ implies that $j^2 f(X) \cap S_{1,r} = \emptyset$ for all $r > 0$.

function is called a Morse function. The classical Morse Lemma states that for a Morse function, f, in a neighborhood of each critical point there exist normal coordinates where f takes the following form:

$$f(x_1, x_2, \ldots, x_m) = c + x_1^2 + x_2^2 + \cdots + x_j^2 - x_{j+1}^2 - \cdots - x_m^2, \tag{2.2.1}$$

where $c = f(0)$ is the value of the critical point, aptly called the *critical value*.

We will want to use this local description of f to decompose X into well understood elementary pieces. One problem that we would run into at this point is that it might be possible for multiple critical points to have the same critical value. We can fix this by using the multi-jet transversality theorem. The multi-jet space has a submersion,

$$\pi : J^{(1,1)}(X, \mathbb{R}) \to \mathbb{R} \times \mathbb{R},$$

so that we may consider the submanifold,

$$S := (S_1 \times S_1) \cap J^{(1,1)}(X, \mathbb{R}) \cap \pi^{-1}(\Delta\mathbb{R}).$$

where $\Delta\mathbb{R} \subset \mathbb{R} \times \mathbb{R}$ is the diagonal. This submanifold has codimension $2 \cdot \dim X + 1$. Thus the multi-jet transversality theorem ensures that for generic f, $j^{(1,1)}f(X^{(2)}) \cap S = \emptyset$. Thus any two critical points of a generic map must have distinct critical values.

Choosing a generic map to the one-dimensional space $\mathbb{R}$ gives us a template for how to decompose the space X in a one-dimensional fashion. We can decompose X into regions where f contains a single isolated critical point and into regions where f has no critical points at all, see Figure 2.1. To completely understand this decomposition of X one must further analyze these pieces. In particular, if $\dim X > 1$, to understand the pieces with isolated critical points, one must choose a generic metric and gradient-like vector field. Only then can one obtain the usual handle structure associated to a Morse function. Moreover, all these pieces are glued together via diffeomorphisms of their respective boundaries, and these diffeomorphisms play an important role in the decomposition.

Instead, we can specialize to the case $\dim X = m = 1$. This simplifies the analysis in two important ways. First of all, metrics and gradient like-vector fields are unnecessary; they give no new information. Secondly, the relevant gluing morphisms are diffeomorphisms of zero-dimensional manifolds and hence are essentially combinatorial. Moreover, the function f, restricted to the regions which contain no critical points, is a local diffeomorphism.

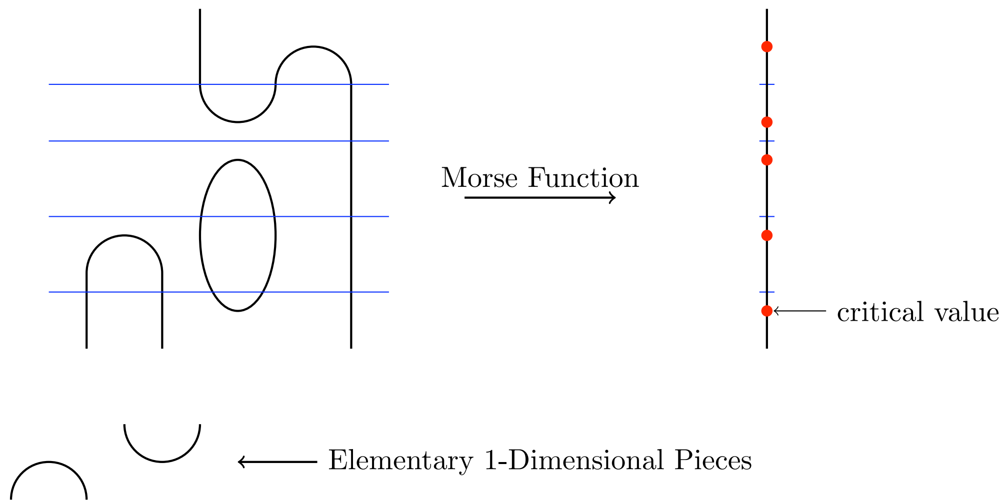


Figure 2.1: A One-Dimensional Decomposition (of a 1-Manfold) Induced by a Morse Function

2.2.2 Rudimentary Cerf Theory

Cerf theory, also called parametrized Morse theory, in its most elementary form is the study of families of Morse functions. There are several ways to motivate the study of such families, but perhaps the simplest is to answer the following question: Given a (compact) manifold X, we know that there exist Morse functions on X and that these induce decompositions of X, but how are these Morse functions related? Is the space of Morse functions is connected? How are the corresponding decompositions related?

The answer to this second question is no, the space of Morse functions is not connected. However, the same techniques which lead to our understanding of Morse functions can be used to answer the remaining questions. We know that the space of all functions $X \to \mathbb{R}$ is connected (in fact, contractible), thus between any two Morse functions we may choose a path of functions. Such a path of functions is the same as a single map $F : X \times I \to \mathbb{R} \times I$, which preserves the projections to I, i.e. we have $F(x,t) = (f_t(x), t)$. By the relative transversality theorem we can deform F to be a a generic function keeping it the same on the ends of I.

Moreover, as we will see, this can be done is such a way that the component $\partial_t pF$ never vanishes, (here $p : \mathbb{R} \times I \to I$ is the projection). So while this deformed F is not

precisely a path, it is quite close and sufficient for our purposes.[2] In fact we can pre-compose F with a diffeomorphism of $X \times I$ to obtain a generic path. We now need only specify a suitable collection of submanifolds of the jet space, with which to make F generic.

What we will find is that such submanifolds can be chosen so that f_t is a Morse function on the complement of a finite number $t \in I$. At these critical times we can again obtain local coordinates, which allow us to completely understand how the corresponding Morse functions change. When X is one-dimensional this allows us to specify a finite list of "moves" or "relations" between Morse decompositions. Any two decompositions are related by a finite sequence of these moves.

The stratification of the jet space we will use is essentially the Thom-Boardman stratification, though we use the presentation due to Porteous [Por71, Por72], see also [GG73]. Some strata will also coincide with the Morin strata. Recall that for any manifolds M, N, we have the identification,

$$ {}_xJ^1(M,N)_y \cong \mathrm{Hom}(T_xM, T_yN). $$

Recall also that for vector spaces V and W, $L^r(V,W)$ is a (natural) submanifold of $\mathrm{Hom}(V,W)$ of dimension $r^2 + r \cdot |\dim V - \dim W|$. It consists of linear maps of corank r. The naturality of this submanifold is such that we can construct canonical submanifolds $S_r \subset J^1(M,N)$ such that the fiber over (x,y) is

$$ S_r := L^r(T_xM, T_yN). $$

Given a linear map $A \in L^r(V,W)$, we can consider the kernel K_A and cokernel L_A which fit into an exact sequence of vector spaces,

$$ 0 \to K_A \to V \xrightarrow{A} W \to L_A \to 0. $$

Over $L^r(V,W)$ the K_A and L_A assemble into (canonical) vector bundles and the normal bundle of $L^r(V,W)$ in $\mathrm{Hom}(V,W)$ at A is canonically isomorphic to $\mathrm{Hom}(K_A, L_A)$, as shown in [GG73, VI § 1]. This induces corresponding vector bundles K and L over S_r. As before, we may obtain a preliminary stratification of $J^2(M,N)$ by pulling back the stratification of

[2] In classical Cerf theory it is proven that we may deform F to be generic, while simultaneously keeping it a path. This is done by stratifying the mapping space $C^\infty(X,Y)$ suitably (although it is not a manifold) and proving that any path between Morse functions may be deformed so as to only pass through the codimension-zero and codimension-one strata.

$J^1(M,N)$. Thus we obtain submanifolds $S_r^{(2)}$. As before, we have an induced map of fiber bundles over S_r,

$$S_r^{(2)} \to \mathrm{Hom}(K \odot K, L)$$

which is a submersion. We look at the inclusion $\mathrm{Hom}(K \odot K, L) \to \mathrm{Hom}(K, \mathrm{Hom}(K, L))$. Now $\mathrm{Hom}(K, \mathrm{Hom}(K, L))$ can be stratified into manifolds $L^s(K, \mathrm{Hom}(K, L))$ and it can be shown that the intersection with $\mathrm{Hom}(K \odot K, L)$ is also a manifold $\mathrm{Hom}(K \odot K, L)_s$, see [GG73]. Thus we may take the inverse image of $\mathrm{Hom}(K \odot K, L)_s$ in $S_r^{(2)}$ to obtain a submanifold $S_{r,s}$. These stratify the 2-jets.

Let us specialize to the case relevant for Morse theory, $M = X \times I$ and $N = \mathbb{R} \times I$. In this case there are only the three strata S_0, S_1, and S_2 of codimensions 0, $\dim X$, and $2 \cdot \dim X + 2$, respectively. Transversality to S_2, thus means that there are no points where the jet of F is in S_2. For points in S_1, both K and L are one-dimensional, so that there are only two cases: $S_{1,1}$ which consists of those elements of $S_1^{(2)}$ lying over the zero section of $\mathrm{Hom}(K \odot K, L)$, and it's complement $S_{1,0}$. The normal bundle to $S_{1,1}$ in $S_1^{(2)}$ is thus canonically isomorphic to $\mathrm{Hom}(K \odot K, L)$, so that $S_{1,1}$ is codimension one in $S_1^{(2)}$, i.e. codimension $\dim X + 1$ in $J^2(X \times I, \mathbb{R} \times I)$.

We may continue this process further. Let $S_{1,1}^{(3)}$ denote the inverse image of $S_{1,1}$ under the projection $J^3(X \times I, \mathbb{R} \times I) \to J^2(X \times I, \mathbb{R} \times I)$. As before, we get a map of fiber bundles over $S_{1,1}$,

$$S_{1,1}^{(3)} \to \mathrm{Hom}(TX, TJ^2(X \times I, \mathbb{R} \times I))$$

which we can project and restrict to get a map to $\mathrm{Hom}(K, \mathrm{Hom}(K \odot K, L))$. As we will see shortly, this actually lands in $\mathrm{Hom}(K \odot K \odot K, L)$, and the composite,

$$S_{1,1}^{(3)} \to \mathrm{Hom}(K \odot K \odot K, L),$$

is a surjective map of fiber bundles over $S_{1,1}$. Again we may stratify $S_{1,1}^{(3)}$ by stratifying $\mathrm{Hom}(K \odot K \odot K, L)$ and pulling the stratification back to $S_{1,1}^{(3)}$. Since all the bundles involved are 1-dimensional, there is an obvious choice for such a stratification. Let $\mathrm{Hom}(K \odot K \odot K, L)_1$ denote the zero section and let $\mathrm{Hom}(K \odot K \odot K, L)_0$ denote its (open) complement. Thus we obtain two strata $S_{1,1,0}$ and $S_{1,1,1}$. The latter has codimension one in $S_{1,1}^{(3)}$ and hence codimension $\dim X + 2$ in $J^3(X \times I, \mathbb{R} \times I)$.

If a map $f : X \times I \to \mathbb{R} \times I$ is such that its jet at (x,t) lands in one of the manifolds we have just constructed, S_α, then we will say that f has a *singularity of type* S_α

at x. The Thom transversality theorem now applies and shows us that the set T_W of maps $f : X \times I \to \mathbb{R} \times I$ whose induced sections $j^k f$ are transverse to $S_0, S_1, S_2, S_{1,0}, S_{1,1}, S_{1,1,0}$ and $S_{1,1,1}$ form a residual (hence dense) subset. In particular, because the codimension of $S_{1,1,1}$ is too high, these kinds of singularities do not occur. Moreover, the points with singularity type S_1 form a 1-dimensional submanifold of $X \times I$, of which there is a 0-dimensional submanifold of $S_{1,1}$ singularities.

We may improve upon this in two ways. First, the submanifolds we have constructed here are compatible with those we used to define Morse functions. Thus if $f_0, f_1 : X \to \mathbb{R}$ are two Morse functions and $F : X \times I \to \mathbb{R} \times I$ is any map satisfying condition (1) of the Relative Transversality Theorem 2.1.7, then F automatically satisfies condition (2), as well. Thus Theorem 2.1.7 applies to this relative case.

Secondly, an arbitrary (generic) map $X \times I \to \mathbb{R} \times I$ is very far from being a path of functions. We can make it closer as follows. Consider the projection of jet bundles,

$$J^1(X \times I, \mathbb{R} \times I) \to J^1(X \times I, I).$$

Inside $J^1(X \times I, I)$ there is a distinguished open subset $\mathcal{O}$, whose inverse image in $J^1(X \times I, \mathbb{R} \times I)$ is also open (we will abuse notation and call this open set $\mathcal{O}$, as well). On $X \times I$ we have a distinguished vector field, which in local coordinates is given by ∂_t, where ∂_t is also the standard vector field on I. The span of this gives us 1-dimensional subbundle $E \subseteq T(X \times I)$, and thus we may consider the projection,

$$J^1(X \times I, I) = T^*(X \times I) \otimes TI \to E^* \otimes TI.$$

The inverse image of the complement of the zero section of $E^* \otimes TI$ gives the open set $\mathcal{O}$. Finally, we may consider the subset of $C^\infty(X \times I, \mathbb{R} \times I)$ given by,

$$M(\mathcal{O}) = \{F \mid j^1F(X \times I) \subset \mathcal{O}\}.$$

This is open in the Whitney C^1-topology, and hence open in the Whitney C^∞-topology. It consists of those maps $F(x,t) = (f_t(x), g(x,t))$ such that the t-derivative of the t^{th}-coordinate (i.e. $\partial_t g$) is non-zero at all points of $X \times I$. A path of functions can be viewed as a map $F : X \times I \to \mathbb{R} \times I$, and such a map is in $M(\mathcal{O})$, so that $M(\mathcal{O})$ is non-empty. Thus we may take our previously constructed residual set T_W and intersect it with $M(\mathcal{O})$. A dense subset of maps in $M(\mathcal{O})$ will also lie in T_W.

To complete our analysis we must derive local coordinates for these singularities and study how these local descriptions combine to tell us about the whole. As before, we will simplify the discussion by considering just the case when X is 1-dimensional. In the process we will also justify our claim that the map

$$S_{1,1}^{(3)} \to \mathrm{Hom}(K, \mathrm{Hom}(K \odot K, L)),$$

lands in $\mathrm{Hom}(K \odot K \odot K, L)$, and is surjective onto its image.

Up to this point, we have primarily been using the geometric description of jet bundles. To accomplish our further analysis we will now make contact with the algebraic description of jet space. Recall that the fibers of the jet bundles can be identified as certain spaces of algebra homomorphisms,

$$ {}_xJ^k(X,Y)_y \cong \mathrm{Hom}(C_y^\infty/\mathfrak{m}_y^{r+1}, C_x^\infty/\mathfrak{m}_x^{r+1}).$$

This algebraic description will allow us to use algebraic techniques to derive normal coordinates around the various singularities we have introduced.

Definition 2.2.2. Let $f : (X,x) \to (Y,y)$ be the germ of a local map taking $x \in X$ to $y \in Y$. The *local ring* of f is the quotient ring: $\mathcal{R}_f = C_x^\infty / C_x^\infty \cdot f^*\mathfrak{m}_y$. This is a local ring with maximal ideal $\mathfrak{m}_f$. If $f : X \to Y$ is an actual function, it induces a map germ at each point $x \in X$, and hence a corresponding local ring, $\mathcal{R}_f(x)$. ◇

The strata of the jet space that we have introduced can alternatively be defined in terms of the corresponding local rings. For example, consider those map germs between X and Y in which the the local ring is $\mathbb{R}$. These are precisely those map germs in which $f^*\mathfrak{m}_y = \mathfrak{m}_x$, and hence $df : T_xX \cong (\mathfrak{m}_x/\mathfrak{m}_x^2)^* \to (\mathfrak{m}_y/\mathfrak{m}_y^2)^* \cong T_yY$ is an isomorphism. These are precisely the map germs whose 1-jet lies in S_0, i.e. S_0 consists of precisely those jets which have local representatives whose local ring is $\mathcal{R}_f \cong \mathbb{R}$. There are similar characterizations for the remaining strata, and in particular having a map germ whose jet is in the strata $S_{1,0}$ or $S_{1,1,0}$ in fact determines the local ring.

The S_r strata have a straightforward description in terms of the local ring. As we have seen, the 1-jets consist of the linear maps:

$$(df)^* : \mathfrak{m}_y/\mathfrak{m}_y^2 \to \mathfrak{m}_x/\mathfrak{m}_x^2.$$

The stratification into sets S_r corresponds to precisely those where the cokernel (or kernel) of this map has dimension r. Thus the S_0 jets are those where (df^*) is an isomorphism (and hence $\mathcal{R}_f \cong \mathbb{R}$). The cokernel of this linear map is the vector space,

$$\tilde{K} := \mathfrak{m}_x/(C^\infty \cdot f^*\mathfrak{m}_y + \mathfrak{m}_x^2) \cong \mathfrak{m}_f/\mathfrak{m}_f^2,$$

and hence is entirely determined by the local ring $\mathcal{R}_f$.

Consider the case were we have a local map germ f, whose 1-jet lies in S_1. We wish to determine whether its 2-jet lies in $S_{1,1}$ or $S_{1,0}$. From our earlier discussion we know that we should be able to extract a linear map,

$$K \odot K \to L,$$

and that if s is the dimension of the cokernel (or kernel) of this map, then the 2-jet is in $S_{1,s}$. What we can construct from the local ring of f is a linear map $L^* \to K^* \odot K^*$, which is essentially equivalent to the above map. Since $\tilde{K}$ is the cokernel of $(df^*) : T_y^*Y \to T_x^*X$, we have $\tilde{K} \cong K^*$. Note that the inverse image of $\mathfrak{m}_x$ under the map $f^* : C_y^\infty \to C_x^\infty$ is precisely the ideal $\mathfrak{m}_y$. Let $\tilde{L}$ be defined by,

$$\tilde{L} := (f^*)^{-1}(\mathfrak{m}_x^2)/\mathfrak{m}_y^2.$$

Thus $\tilde{L}$ is precisely the kernel of $(df)^*$, and hence $\tilde{L} \cong L^*$.

Proposition 2.2.3. *The multiplication map induces an isomorphism of vector spaces,*

$$\tilde{K} \odot \tilde{K} = \frac{\mathfrak{m}_x}{(C^\infty \cdot f^*\mathfrak{m}_y + \mathfrak{m}_x^2)} \odot \frac{\mathfrak{m}_x}{(C^\infty \cdot f^*\mathfrak{m}_y + \mathfrak{m}_x^2)} \xrightarrow{\mu} \frac{\mathfrak{m}_x^2}{(\mathfrak{m}_x \cdot f^*\mathfrak{m}_y + \mathfrak{m}_x^3)}.$$

Proof. Choosing local coordinates, it is easy to check that the following square commutes:

$$\begin{tikzcd}
(\mathfrak{m}_x/\mathfrak{m}_x^2) \odot (\mathfrak{m}_x/\mathfrak{m}_x^2) \arrow[r, "\mu"] & \mathfrak{m}_x^2/\mathfrak{m}_x^3 \\
(C^\infty \cdot f^*\mathfrak{m}_y/\mathfrak{m}_x^2) \odot (\mathfrak{m}_x/\mathfrak{m}_x^2) \arrow[u, hook] \arrow[r, "\mu"'] & (\mathfrak{m}_x \cdot f^*\mathfrak{m}_y)/(\mathfrak{m}_x^2 \cdot f^*\mathfrak{m}_y) \arrow[u, hook]
\end{tikzcd}$$

Here the symmetrization $C^\infty \cdot f^*m_y \odot \mathfrak{m}_x$ makes sense since $C^\infty \cdot f^*\mathfrak{m}_y \subseteq \mathfrak{m}_x$. Moreover, the kernel of the top map is clearly zero. A dimension count then shows that both maps in the square labeled μ are isomorphisms, and hence the induced map between the cokernels of the vertical maps is also an isomorphism. □

The elements of $\tilde{L}$ are equivalence classes of elements of $\mathfrak{m}_y$ which happen to land in $\mathfrak{m}_x^2$. Thus we have a natural map,

$$\tilde{L} \to \frac{\mathfrak{m}_x^2}{(\mathfrak{m}_x \cdot f^*\mathfrak{m}_y + \mathfrak{m}_x^3)} \cong \tilde{K} \odot \tilde{K}. \tag{2.2.4}$$

In fact this map is the transpose of the map $K \odot K \to L$ considered earlier. We can see this by choosing coordinates (x_0, x_1) around x and (y_0, y_1) around y such that df has the form

$$df_x = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus $f_0 = ax_0^2 + bx_0x_1 + cx_1^2 + O(|x|^3)$. The above map is then given precisely by the coefficient a, and the map $K \odot K \to L$ is given by (a scalar multiple of) the same coefficient.

Thus we may characterize the $S_{1,s}$ jets in terms of the local ring $\mathcal{R}_f$. They are precisely those in which the map in equation 2.2.4 has a cokernel (or kernel) of dimension s. The cokernel is the space,

$$\frac{\mathfrak{m}_x^2}{(\mathfrak{m}_x^2 \cap (C_x^\infty \cdot f^*m_y) + \mathfrak{m}_x^3)} \cong \mathfrak{m}_f^2/\mathfrak{m}_f^3,$$

and hence is completely determined by the local ring. Moreover, we see also that the only local ring which induces an $S_{1,0}$ jet is $\mathcal{R}_f \cong \mathbb{R}[x]/(x^2)$.

Now we can analyze those local rings which correspond to $S_{1,1,s}$-jets. To characterize these we will need to extract from the local ring a map,

$$\tilde{L} \to \tilde{K} \odot \tilde{K} \odot \tilde{K}. \tag{2.2.5}$$

A similar calculation as before shows that,

$$\tilde{K} \odot \tilde{K} \odot \tilde{K} \cong \frac{\mathfrak{m}_x^3}{(\mathfrak{m}_x^2 \cdot f^*\mathfrak{m}_y + \mathfrak{m}_x^4)},$$

via the multiplication map. Moreover, we are already assuming that our local map germ induces a jet in $S_{1,1}$. Thus the map from $\tilde{L}$ to $\tilde{K} \odot \tilde{K}$ is the zero map, and hence elements of $\tilde{L}$ are equivalence classes of elements of $\mathfrak{m}_y$ whose image lies in $(\mathfrak{m}_x \cdot f^*\mathfrak{m}_y + \mathfrak{m}_x^3)$. This has two consequences. First, since $\tilde{L}$ is also represented by equivalence classes of elements of $\mathfrak{m}_y$ which land in $\mathfrak{m}_x^2$, we know that the image of such an element is either already in $\mathfrak{m}_x^3$, in $\mathfrak{m}_x \cdot f^*\mathfrak{m}_y$, or in a linear combination of the two. In particular those elements of the form $\mathfrak{m}_x \cdot f^*\mathfrak{m}_y \cap \mathfrak{m}_x^3$ actually lie in $(\mathfrak{m}_x^2 \cdot f^*\mathfrak{m}_y + \mathfrak{m}_x^4)$. Secondly we have a naturally induced map,

$$\tilde{L} \to \frac{\mathfrak{m}_x \cdot f^*\mathfrak{m}_y + \mathfrak{m}_x^3}{(\mathfrak{m}_x \cdot f^*\mathfrak{m}_y + \mathfrak{m}_x^4)} \cong \frac{\mathfrak{m}_x^3}{(\mathfrak{m}_x \cdot f^*\mathfrak{m}_y \cap \mathfrak{m}_x^3 + \mathfrak{m}_x^4)} \cong \frac{\mathfrak{m}_x^3}{\mathfrak{m}_x^2 \cdot f^*\mathfrak{m}_y + \mathfrak{m}_x^4}.$$

This is precisely the map we needed in Equation 2.2.5.

Working in local coordinates again, we see that f_0 has the form,

$$f_0 = ax_0x_1 + bx_1^2 + cx_0^3 + dx_0^2x_1 + ex_0x_1^2 + fx_1^3 + O(|x|^4).$$

The above map (which is a map between 1-dimensional vector spaces) is given by the coefficient c (which must be non-zero for $S_{1,1,0}$). This map does indeed agree (at least up to a non-zero scalar) with the transpose of the map $K \odot K \odot K \to L$ used to define $S_{1,1,s}$. Clearly there exist f obtaining any value of c that one wishes, hence the map

$$S_{1,1}^{(3)} \to \mathrm{Hom}(K \otimes K \otimes K, L)$$

is surjective, as claimed earlier. Thus those map germs with induced jets in $S_{1,1,s}$ are precisely those for which the above map has a cokernel (or kernel) of dimension s. The cokernel of this map is the space,

$$\frac{\mathfrak{m}_x^3}{C_x^\infty \cdot f^*\mathfrak{m}_y \cap \mathfrak{m}_x^3} \cong \mathfrak{m}_f^3/\mathfrak{m}_f^4$$

so once again this is completely determined by the local ring $\mathcal{R}_f$, and moreover the only local ring which induces an $S_{1,1,0}$ singularity is $\mathbb{R}_f \cong \mathbb{R}[x]/(x^3)$.

The reason for passing to the algebraic description of jet space and the singularities S_α is that we now have many powerful algebraic tools at our disposal. This will be extremely helpful in deriving the normal coordinates associated to these singularities, especially in our later applications. The main theorem that we will use is the Generalized Malgrange Preparation Theorem and its corollaries. Proofs of these classical theorems may be found, for example, in [GG73, Chapter IV].

Theorem 2.2.6 (Generalized Malgrange Preparation Theorem)**.** *Let X and Y be smooth manifolds and $\phi : X \to Y$ be a smooth mapping with $\phi(p) = q$. Let A be a finitely generated $C_p^\infty(X)$-module. Then A is a finitely generated $C_q^\infty(Y)$-module if and only if $A/\mathfrak{m}_q(Y) \cdot A$ is a finite dimensional vector space over $\mathbb{R}$.*

Corollary 2.2.7. *If the projections of $e_1, \ldots, e_k$ form a spanning set of vectors in the vector space $A/(\mathfrak{m}_p^{k+1}(X) \cdot A + \mathfrak{m}_q(Y) \cdot A)$ then $e_1, \ldots, e_k$ form a set of generators for A as a $C_q^\infty(Y)$-module.*

Let $f : X^1 \times I \to \mathbb{R} \times I$ be a generic function (i.e. transverse to S_i, $S_{i,j}$ and $S_{i,j,k}$ for all admissible i, j, k). The the singularities of f must be one of the following three types: S_0,

$S_{1,0}$ and $S_{1,1,0}$. These occur with codimensions 0, 1, and 2 respectively. The singularities S_0 are not actually singularities at all; they correspond to points where the differential *df* is invertible, and so by the inverse function theorem, f is a local diffeomorphism at these points. The derivation of the local coordinates for these singularities is a well known classical result. We will only review the $S_{1,1,0}$ case as a means to demonstrate our use of these powerful algebraic theorems. The $S_{1,0}$ case can be handled in a similar fashion.

The $S_{1,1,0}$ critical points are isolated in $X^1 \times I$. Let $p \in X \times I$, be such a critical point. The differential *df* has rank one at the point p and so by the implicit function theorem, together with our assumptions on the t-derivatives of $\pi \circ f$, imply that there exist local coordinates (x_0, x_1) for $X \times I$ centered at p such that f has the following form

$$f^* y = h(x_0, x_1)$$
$$f^* t = x_1$$

(here $\pi : \mathbb{R} \times I \to I$ is the projection). Moreover since we are at a $S_{1,1,0}$ singularity, h has an expansion,

$$h(x_0, x_1) = \alpha_{01} x_0 x_1 + \alpha_{11} x_1^2 + \beta_{000} x_0^3 + \beta_{001} x_0^2 x_1 + \beta_{011} x_0 x_1^2 + \beta_{111} x_1^3 + O(|x|^4)$$

in which β_{000} is non-zero. In fact, by assumption $j^2 f$ is transverse to $S_{1,1}$, which is equivalent to the statement that the following map of vector spaces,

$$T_p X \to T_{j^2 f(p)} J^2(X \times I, \mathbb{R} \times I)/T_{j^2 f(p)} S_{1,1} \cong \nu S_{1,1} \cong \mathrm{Hom}(K, L) \oplus \mathrm{Hom}(K \odot K, L),$$

is an isomorphism. In these coordinates, this map is given by the matrix,

$$\begin{pmatrix} 0 & \alpha_{01} \\ \beta_{000} & \beta_{001} \end{pmatrix}.$$

In particular the coefficient α_{01} must be non-zero.

The local ring $\mathcal{R}_f(p)$ is isomorphic to $\mathbb{R}[x_0]/(x_0^3)$, and so by the Generalized Malgrange Preparation Theorem 2.2.6, and more specifically Corollary 2.2.7, we know that there exist smooth functions a, b, c on $\mathbb{R} \times I$ such that,

$$x_0^3 = f^* a + f^* b \cdot x_0 + f^* c \cdot x_0^2.$$

Moreover, by replacing x_0 by the coordinate $x_0 + \frac{1}{2} f^* c$ (and keeping all other coordinates the same) we may assume that $c = 0$. Expanding both sides of Equation 2.2.2 when $x_1 = t = 0$

and identifying the lowest order terms we find that

$$\frac{\partial a}{\partial y}(p) = \frac{1}{\beta_{000}} \neq 0$$

Thus the following is a valid change of coordinates,

$$\begin{aligned}\overline{y} &= a(y,t)\\ \bar{t} &= t\end{aligned}$$

and in these coordinates Equation 2.2.2 becomes

$$x_0^3 = f^*y + f^*b \cdot x_0.$$

Now expanding Equation 2.2.2 in both x_0 and x_1 and gathering the x_1x_0 terms, we see that

$$\frac{\partial b}{\partial t}(q) = -\frac{\alpha_{01}}{\beta_{000}} \neq 0.$$

Thus the following is a valid pair of coordinate changes,

$$\begin{aligned}\overline{y}_0 &= y\\ \overline{y}_1 &= b(y,t)\\ \overline{x}_0 &= x_0\\ \overline{x}_1 &= f^*b.\end{aligned}$$

Note: depending on the signs of α_{01} and β_{000} this may be an orientation reversing coordinate change. These results may be summarized by the following proposition.

Proposition 2.2.8. *Let $f : X^1 \times I \to \mathbb{R} \times I$ be a generic function and let $p \in X$ be a point with an $S_{1,1,0}$ singularity. Then there exist local coordinates (x_0, x_1) of $X \times I$ centered at p and coordinates (y_0, y_1) of $Y \times I$ centered at $q = f(p)$ such that f has the following form:*

$$\begin{aligned}f^*y_0 &= x_0^3 + x_1x_0\\ f^*y_1 &= x_1.\end{aligned}$$

A similar analysis applies and gives local coordinates for the $S_{1,0}$ singularities. One may then use these local coordinates to get a global description of generic functions $F : X \times I \to \mathbb{R} \times I$. Any two morse functions may be extended to such a generic function F and each of the singularities presented has an interpretation in terms of the critical points of

the Morse function. The S_0 points correspond to the regular values of the Morse function, the $S_{1,0}$ singularities correspond to paths of critical points moving around in X, and the $S_{1,1,0}$ singularities correspond to birth-death critical point cancelations. One may use this description to prove that any two Morse functions on a (compact) 1-manifold X are related by a path of functions which consists entirely of Morse functions except at a finite number of times, in which a birth-death occurs. We will not pursue this here. Instead we now turn to higher dimensional decompositions.

2.2.3 Strategy of the Planar Decomposition Theorem

We have seen how a generic map from a manifold X to $\mathbb{R}$ induces a 1-dimensional decomposition of the manifold. Similarly a generic map from Σ to $\mathbb{R}^2$ will induce a 2-dimeensional decomposition of X. Just as studying certain generic maps $X \times I \to \mathbb{R} \times I$ allows us to compare different 1-dimensional decompositions, studying generic maps $\Sigma \times I \to \mathbb{R}^2 \times I$ will allow use to compare different 2-dimensional decompositions.

Later we will want to use these planar decompositions of Σ to extract 2-categorical information. For this purpose it is also useful to control not just the map $\Sigma \to \mathbb{R}^2$, but also the composition with the projection to $\mathbb{R}$, a map $\Sigma \to \mathbb{R}$. This is similar to the requirement that we imposed on our maps $X \times I \to \mathbb{R} \times I$. We required that, after composing with the projection to I, the map $X \times I \to I$ satisfies certain properties. When we look at maps $\Sigma \times I \to \mathbb{R}^2 \times I$, we will also want to control the projections to $\mathbb{R} \times I$ and to I. Incorporating this control is in principle straightforward. We end up with a stratification of the jet bundles which is slightly more refined then the classical Thom-Boardman stratification considered in the previous sections.

After giving this stratification, we will give an analysis similar to the previous one, deriving normal coordinates for these singularities. For the case of maps $\Sigma^2 \to \mathbb{R}^2$, this is nearly identical to the derivation of the last section. In the 3-dimensional case some non-standard decompositions appear, but the method of their derivation is not essentially different then the classical case. We will also deal more carefully with multi-jet issues, which were completely ignored in our previous discussion.

The end result of these considerations will be that a generic map $\Sigma^2 \to \mathbb{R}^2$ will induce a planar decomposition of Σ into elementary pieces. Two such generic maps can then by connected by a generic map $\Sigma^2 \times I \to \mathbb{R}^2 \times I$, hence two such planar decomposi-

tions are related if there is a decomposition of $\Sigma^2 \times I$ into a 3-dimesnional arrangement of elementary pieces. These give "relations" or "moves" connecting any two planar decompositions. Finally we will discuss the minor changes necessary to adapt these results to the case where Σ has boundary and corners. For simplicity, in what follows we will restrict our attention to the case where Σ^2 is a 2-dimensional manifold. The general situation requires a more nuanced discussion involving the choice of gradient like vector fields and similar accoutrements.

2.2.4 The Stratification of the Jet Space: 2D case

The stratification of the jet space that we will use for maps $\Sigma^2 \to \mathbb{R}^2$ is very similar to the Thom-Boardman stratification we used to study maps $X^1 \times I \to \mathbb{R} \times I$. In fact two of our strata, S_0 and S_2 of codimensions zero and four, respectively, are defined exactly as they were for the Thom-Boardman stratification. The third Thom-Boardman stratum, S_1, will be spilt into two separate strata $S_{[01]}$ and $S_{[11]}$ of codimensions one and two, respectively. This finer stratification results from our consideration of the composition $\Sigma^2 \to \mathbb{R}^2 \to \mathbb{R}^1$. These strata then induce preliminary stratifications of the higher jets, just as before, and, also just as before, we may further refine these strata in a straightforward way. All in all, we have the strata listed in Table 2.1, grouped according to which jet space they reside in, together with their respective codimensions.

Table 2.1: The Stratification of Jet Space: 2D Case

Strata	Codimension	Name
S_0	0	Local Diffeomorphism
$S_{[01]}$	1	·
$S_{[11]}$	2	·
S_2	4	n/a
$S_{[01],0}$	1	Fold
$S_{[01],1}$	2	·
$S_{[11],0}$	2	2D Morse
$S_{[01],1}$	3	·
$S_{[01],1,0}$	2	Cusp
$S_{[01],1,1}$	3	·

Based off these codimensions, a generic map $f : \Sigma \to \mathbb{R}^2$ only admits singularities of types $S_0, S_{[0,1],0}, S_{[11],0}$, and $S_{[0,1],1,0}$. We take up the task of deriving normal coordinates for these in the next section. Recall that the Thom-Boardman stratum S_1 of $J^1(\Sigma, \mathbb{R}^2)$ was defined as the subset of jets, viewed as elements of $\mathrm{Hom}(T_x\Sigma, T_y\mathbb{R}^2)$, having corank one (and hence also rank one). The projection $p : \mathbb{R}^2 \to \mathbb{R}$ induces a projection,

$$J^1(\Sigma, \mathbb{R}^2) \to J^1(\Sigma, \mathbb{R})$$

which on fibers is the projection $\mathrm{Hom}(T_x\Sigma, T_y\mathbb{R}^2) \to \mathrm{Hom}(T_x\Sigma, T_{p(y)}\mathbb{R})$. This later space can also be stratified by corank and pulling this stratification back induces a refined stratification of S_1.[3] There are two such strata corresponding to when $d(pf)$ has corank zero and corank one (rank one and rank zero, respectively). This defines the strata $S_{[s\ t]}$ which consists of jets where $d(pf)_x$ has corank s and df_x has corank t.

We see that $S_{[01]}$ is an open subset of S_1. Let $S^{(2)}_{[01]}$ and $S^{(3)}_{[01]}$ denote the inverse image of $S_{[01]}$ in $J^2(\Sigma, \mathbb{R}^2)$ and $J^3(\Sigma, \mathbb{R}^2)$ under the respective projections. We define,

$$\begin{aligned} S_{[01],r} &:= S^{(2)}_{[01]} \cap S_{1,r}, \\ S_{[01],r,s} &:= S^{(3)}_{[01]} \cap S_{1,r,s}. \end{aligned}$$

These are manifolds since $S_{[01]}$ is open in S_1. $S_{[11]}$ has codimension 2 in the jet space $J^1(\Sigma, \mathbb{R}^2)$ and its normal bundle is canonically isomorphic to $\mathrm{Hom}(T\Sigma, T\mathbb{R})$ (where these bundles are interpreted as their pull-back to $S_{[11]}$. Over S_1 there are two canonical bundles, K and L, which we introduced previously, and these restrict to bundles over $S_{[11]}$. Let $S^{(2)}_{[11]}$ denote the inverse image of $S_{[11]}$ in $J^2(\Sigma, \mathbb{R}^2)$. As before we have a map of fiber bundles over $S_{[11]}$,

$$S^{(2)}_{[11]} \to \mathrm{Hom}(T\Sigma, TJ^1(\Sigma, \mathbb{R}^2)) \to \mathrm{Hom}(K, \mathrm{Hom}(K, L)).$$

This composition lands in $\mathrm{Hom}(K \odot K, L)$ and is surjective, as can be checked by choosing local coordinates. Our stratification of $\mathrm{Hom}(K \odot K, L)$ into the two strata $\mathrm{Hom}(K \odot K, L)_0$ and $\mathrm{Hom}(K \odot K, L)_1$ induces a refinement of $S^{(2)}_{[11]}$ into the strata $S_{[11],0}$ and $S_{[11],1}$ respectively. These have codimensions 2 and 3 in $J^2(\Sigma, \mathbb{R}^2)$, respectively. This completes our stratification of the jet space in the 2-dimensional case.

[3]That $S_{[01]}$ is a manifold is clear. The fact that $S_{[11]}$ is also a manifold can be deduced by looking at local coordinates for a representative map germ.

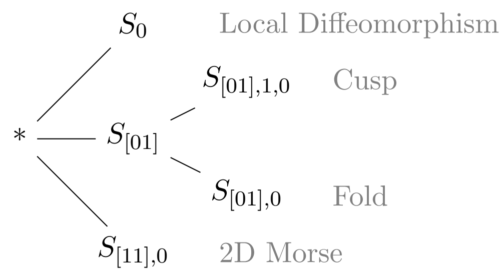


Figure 2.2: The 2D Singularity Family Tree

2.2.5 Normal Coordinates: 2D Case

We must derive normal coordinates for the four kinds singularities of type S_0, $S_{[0,1],0}$, $S_{[11],0}$, and $S_{[0,1],1,0}$. The S_0 points consist of local diffeomorphisms, as already seen. Since we are making use of the projection $\mathbb{R}^2 \to \mathbb{R}$, it is natural for us to find coordinates for $\mathbb{R}^2$ which are compatible with this projection. This is not always possible, however we will always be able to find coordinates $(\overline{y}_0, \overline{y}_1)$ in which the kernel of the differential of the projection at $q \in \mathbb{R}^2$ is the span of $\partial_{\overline{y}_0}$. This is depicted in Figure 2.3. We will also make

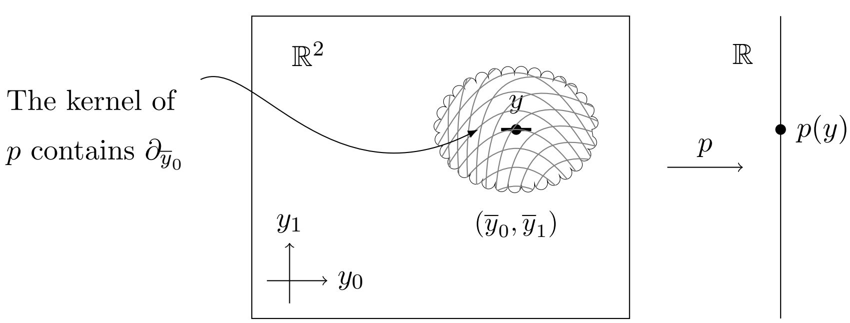


Figure 2.3: The Standard Coordinates for $\mathbb{R}^2$ and its projection to $\mathbb{R}$

the convention that (y_0, y_1) denote the standard coordinates in $\mathbb{R}^2$ and that the projection is given by $p(y_0, y_1) = y_1$. For later purposes, it will also be convenient to keep track of whether the coordinates $(\overline{y}_0, \overline{y}_1)$ preserve the standard orientation on $\mathbb{R}^2$, and whether the orientation on $\mathbb{R}$ induced by the image of $\partial_{\overline{y}_1}$ agrees with the standard orientation of $\mathbb{R}$. If this is the case, then we say $(\overline{y}_0, \overline{y}_1)$ preserves both the orientations of $\mathbb{R}$ and $\mathbb{R}^2$.

Proposition 2.2.9 (Cusp). *Let $f : \Sigma^2 \to \mathbb{R}^2$ be generic and let $x \in \Sigma$ be a critical point*

with singularity type $S_{[01],1,0}$. Then there exist coordinates $(\overline{x}_0, \overline{x}_1)$ of Σ centered at x and coordinates $(\overline{y}_0, \overline{y}_1)$ of $\mathbb{R}^2$ centered at $y = f(x)$ such that $\partial_{y_0} \in \ker p$ at y, such that in these coordinates f takes the normal form:

$$f^*\overline{y}_0 = \overline{x}_0^3 \pm \overline{x}_1\overline{x}_0$$
$$f^*\overline{y}_1 = \overline{x}_1.$$

Moreover, the coordinates $(\overline{y}_0, \overline{y}_1)$ can be chosen so as to agree with the orientations of $\mathbb{R}^2$ and $\mathbb{R}$ at y and $p(y)$.

Proof. This follows essentially from the calculations of the previous section. The differential df has rank one at the point $x \in \Sigma$ and the differential $d(pf)$ also has rank one. Thus by the implicit function theorem there exist local coordinates (x_0, x_1) for Σ centered at x such that f has the following form:

$$f^*y_0 = h(x_0, x_1)$$
$$f^*y_1 = x_1$$

Moreover, h has an expansion of the form,

$$h(x_0, x_1) = \sum_{0 \le i \le j \le 1} \alpha_{ij} x_i x_j + \sum_{0 \le i \le j \le k \le 1} \beta_{ijk} x_i x_j x_k + O(|x|^4).$$

Since x is an $S_{[01],1,0}$ we know that $\alpha_{00} = 0$ and that $\beta_{000} \neq 0$. Morevoer, since $j^3 f \pitchfork S_{[01],1,0}$, the matrix,

$$\begin{pmatrix} 0 & \alpha_{01} \\ \beta_{000} & \beta_{001} \end{pmatrix}.$$

is non-degenerate, which is equivalent to $\alpha_{01} \neq 0$ (together with the already satisfied condition, $\beta_{000} \neq 0$). As before the local ring is isomorphic to $\mathcal{R}_f(x) = \mathbb{R}[x_0]/(x_0^3)$, so by the Malgrange Preparation Theorem 2.2.6, there are functions a, b, c on $\mathbb{R}^2$ such that

$$x_0^3 = f^*a + f^*b \cdot x_0 + f^*c \cdot x_0^2.$$

The same coordinate change in the domain Σ that was used last time allows us to assume that $c \equiv 0$. By expanding this the variables and collecting similar terms we find that,

$$\frac{\partial a}{\partial y_0}(y) \neq 0, \qquad \frac{\partial a}{\partial y_1}(y) = 0, \qquad \frac{\partial b}{\partial y_1}(y) \neq 0.$$

Thus we find that the following is a valid pair of coordinate changes, and that at y $\partial_{\overline{y}_0}$ is proportional to ∂y_0,

$$\begin{aligned}\overline{y}_0 &= \pm a(y_0, y_1)\\ \overline{y}_1 &= \pm b(y_0, y_1)\\ \overline{x}_0 &= \pm x_0\\ \overline{x}_1 &= \pm f^* b.\end{aligned}$$

Choosing the first two signs in the above coordinate change correctly we may arrange for the desired property regarding orientations, and adjusting the sign of $\pm x_0$ appropriately ensures that in these coordinates f has the desired normal form. □

Proposition 2.2.10 (2D Morse). *Let $f : \Sigma^2 \to \mathbb{R}^2$ be generic and let $x \in \Sigma$ be a critical point with singularity type $S_{[11],0}$. Then there exist coordinates $(\overline{x}_0, \overline{x}_1)$ of Σ centered at x, and coordinates $(\overline{y}_0)$ of $\mathbb{R}$ centered at $p(y) = pf(x)$ (thus $(\overline{y}_0, y_1)$ form coordinates for $\mathbb{R}^2$, with y_1 standard), such that f takes the following normal form,*

$$\begin{aligned}f^*\overline{y}_0 &= \overline{x}_0\\ f^* y_1 &= \pm \overline{x}_0^2 \pm \overline{x}_1^2.\end{aligned}$$

Moreover, these coordinates preserve the orientations of $\mathbb{R}^2$ and $\mathbb{R}$.

Proof. The differential df has rank one at the point $x \in \Sigma$ and the differential $d(pf)$ has rank zero. Thus by the implicit function theorem there exist local coordinates (x_0, x_1) for Σ centered at x such that f has the following form:

$$\begin{aligned}f^* y_0 &= x_0\\ f^* y_1 &= h(x_0, x_1).\end{aligned}$$

Since $j^2 f \pitchfork S_{[11],0}$ we know that the map,

$$T_x X \to \mathrm{Hom}(T_x X, T_{p(y)}\mathbb{R}) \cong (\nu S_{[11],0})_{j^2 f(x)},$$

is a surjection. This implies that the Hessian of h is non-degenerate. The expansion of h has the form,

$$h(x_0, x_1) = \sum_{0 \le i \le j \le 1} \alpha_{ij} x_i x_j + O(|x|^3),$$

and our conditions that x is an $S_{[11],0}$ singularity ensure that $\alpha_{11} \neq 0$.

Let $K = -\alpha_{01}/\alpha_{00}$. The following coordinate change is valid:

$$\tilde{x}_0 = x_0$$
$$\tilde{x}_1 = x_1 - Kx_0$$

and in these coordinates we have (removing the tildes),

$$h = \alpha_{11} \cdot x_1^2 + \beta \cdot x_0^2 + O(|x|^3),$$

where $\beta \neq 0$. Thus for some functions $a(x_0)$, $b(x_0)$, and $c(x_0, x_1)$, such that $a(0) = \beta \neq 0$ and $c(0,0) = \alpha_{11} \neq 0$, we have

$$h = a(x_0) \cdot x_0^2 + 2b(x_0) \cdot x_0^2 x_1 + c(x_0, x_1) \cdot x_1^2.$$

Now we can make the coordinate change,

$$\tilde{x}_0 = x_0$$
$$\tilde{x}_1 = x_1 + \frac{b(x_0)}{\alpha_{11}} \cdot x_0^2$$

under which we have (again removing tildes),

$$h = \tilde{a}(x_0) \cdot x_0^2 + \tilde{c}(x_0, x_1) \cdot x_1^2,$$

for some new functions $\tilde{a}$ and $\tilde{c}$ such that $\tilde{a}(0) = \beta \neq 0$ and $c(0,0) = \alpha_{11} \neq 0$. Finally we form the following valid pair of coordinate changes,

$$\overline{x}_0 = x_0 \sqrt{\tilde{a}(x_0)}$$
$$\overline{x}_1 = x_1 \sqrt{\tilde{c}(x_0, x_1)}$$
$$\overline{y}_0 = y_0 \sqrt{\tilde{a}(y_0)}$$
$$\overline{y}_1 = y_1$$

at least in a sufficiently small neighborhood of x and y. In these coordinates f has the claimed normal form. □

Let $S_{[01],0}(f) = Y$ denote the set of elements of Σ which have $S_{[01],0}(f)$ singularities. By transversality, we know that Y is an embedded 1-dimensional submanifold of Σ^2. The conditions the we used to define the stratum $S_{[01],0}(f)$ ensure that both $d(f|_Y)$ and $d(pf|_Y)$

are non-degenerate. In particular, pf is a local diffeomorphism and the map $f : Y \to \mathbb{R}^2$ is (locally) an embedding. In fact (locally) its image is the graph $\{(\gamma(y_1), y_1)\}$ of a function $\gamma : \mathbb{R} \to \mathbb{R}$. These considerations give the first half of the following proposition.

Proposition 2.2.11 (Folds). *Let $f : \Sigma^2 \to \mathbb{R}^2$ be generic and let $x \in \Sigma$ be an $S_{[01],0}$ singularity point. Then there exists a submanifold $Y^1 \subset \Sigma$ containing x and consisting entirely of $S_{[01],0}$ singularities, a neighborhood $U \subset \mathbb{R}$ of $p(y) = pf(x)$, and a function $\gamma : U \to \mathbb{R}$ whose graph $\{(\gamma(y_1), y_1) \,|y_1 \in U\}$ consists precisely of the image of Y under f. Moreover there exist coordinates (x_0, x_1) for Σ centered at x, such that in these coordinates $Y = \{x_0 = 0\}$ (the x_1-axis) and such that f has the following normal form:*

$$\begin{aligned} f^* y_0 &= \pm \overline{x_0}^2 + \gamma(x_1) \\ f^* y_1 &= x_1. \end{aligned}$$

where (y_0, y_1) are the standard coordinated for $\mathbb{R}^2$.

Proof. The existence of Y, U, and γ as in the statement of the proposition have already been established. By the implicit function theorem we may choose coordinates (x_0, x_1) for Σ centered at x, such that $Y = \{x_0 = 0\}$, and such that in these coordinates

$$\begin{aligned} f^* y_0 &= h(x_0, x_1) \\ f^* y_1 &= x_1. \end{aligned}$$

As before, we can expand h as,

$$h(x_0, x_1) = \sum_{0 \le i \le j \le 1} \alpha_{ij} \cdot x_i x_j + O(|x|)^3$$

and our assumption that the points of Y are $S_{[01],0}$ singularity points implies that $h(0, x_1) = \gamma(x_1)$ and that

$$\frac{\partial h}{\partial x_0}(0, x_1) \equiv 0.$$

We temporarily consider the new function $g(x_0, x_1) = h(x_0, x_1) - \gamma(x_1)$. We have $\frac{\partial g}{\partial x_0}(0, x_1) \equiv 0$ and $g(x_0, x_1) \equiv 0$. Thus we may write

$$h(x_0, x_1) = x_0^2 \tilde{g}(x_0, x_1) + \gamma(x_1)$$

for some new function $\tilde{g}$. Our transversality assumptions imply that $\tilde{g}(0, x_1)$ is a non-vanishing function, and so the following is a valid coordinate change:

$$\begin{aligned} \overline{x}_0 &= x_0\sqrt{\tilde{g}(x_0, x_1)} \\ \overline{x}_1 &= x_1, \end{aligned}$$

at least in a sufficiently small neighborhood of Y. In these coordinates f has the desired form. □

2.2.6 The Stratification of the Jet Space: 3D Case

We will also want to fully understand generic maps from $\Sigma^2 \times I$ to $\mathbb{R}^2 \times I$ in a similar manner to our understanding of generic maps for Σ to $\mathbb{R}^2$. As before, we will accomplish this by judiciously choosing a stratification of the jet spaces $J^k(\Sigma \times I, \mathbb{R}^2 \times I)$, and again we will want to leverage the fact that we have projections $p : \mathbb{R}^2 \times I \to \mathbb{R} \times I$ and $p : \mathbb{R} \times I \to I$. Due to the higher dimensionality, our stratification is necessarily more complicated than the previous stratification, but the method which we use to construct the stratification is, in principle, the same. Ignoring multi-jet issues, we will require that our generic maps induce sections $j^k f$ which are transverse to the 28 strata listed in Table 2.2, which lists the strata and their codimensions, grouped according to which jet space they reside in.

Based on this stratification and the respective codimensions, we see that a generic map $f : \Sigma \times I \to \mathbb{R}^2 \times I$ will only have singularities of the nine types labeled with names in Table 2.2. Among these, we will be able to eliminate the possibility of "3D-Morse" singularities ($S_{[111],0}$) by restricting ourselves to a suitable open subset of maps f. To complete our analysis, we will need to derive local coordinates for these eight remaining singularity types. This is taken up in the following section.

For these dimensions (a 3-manifold mapping to a 3-manifold) the classical Thom-Boardman stratification divides the first jet space into four submanifolds, S_r with $r = 0, 1, 2, 3$, based on the corank r of the differential df. Using our projections, p, we can further refine the strata S_1 and S_2, however since S_2 is already of codimension four, we will only need to refine S_1. Just as for the 2-dimensional stratification, we will compare the relative coranks of all the differentials df, $d(pf)$ and $d(p^2 f)$. In this case we see that there are three possibilities, which are indexed by $[r\ s\ t]$, where r is the corank of $d(p^2 f)$, s is the

Table 2.2: The Stratification of Jet Space: 3D Case

Strata	Codimension	Name
S_0	0	Local Diffeomorphism
S_1	1	·
$S_{[001]}$	1	·
$S_{[011]}$	2	·
$S_{[111]}$	3	·
S_2	4	·
S_3	9	·
$S_{[001],0}$	1	1D Morse (Fold)
$S_{[001],[01]}$	2	·
$S_{[001],[11]}$	3	·
$S_{[011],[000]}$	2	2D Morse
$S_{[011],[001]}$	3	·
$S_{[011],[100]}$	3	·
$S_{[011],[101]}$	4	·
$S_{[011],[111]}$	5	·
$S_{[011],[211]}$	6	·
$S_{[111],0}$	3	3D Morse
$S_{[111],1}$	4	n/a
$S_{[001],[01],0}$	2	1D Morse Relation (Cusp)
$S_{[001],[01],1}$	3	·
$S_{[001],[11],0}$	3	Cusp Inversion
$S_{[001],[11],1}$	4	·
$S_{[011],[001],0}$	3	Cusp Flip
$S_{[011],[001],1}$	4	·
$S_{[011],[100],0}$	3	2D Morse Relation
$S_{[011],[100],1}$	4	·
$S_{[001],[01],1,0}$	3	Swallowtail
$S_{[001],[01],1,1}$	4	·

corank of $d(pf)$, and t is the corank of df (which is 1 in all cases). The proof that these are manifolds is completely analogous to the previous 2-dimensional case.

Let $S_\alpha^{(2)}$ denote the inverse image of S_α in $J^2(\Sigma \times I, \mathbb{R}^2 \times I)$ under the natural projection to $J^1(\Sigma \times I, \mathbb{R}^2 \times I)$. The classical Thom-Boardman stratification of the second jet space spilts $S_1^{(2)}$ into two strata $S_{1,1}$ and $S_{1,0}$. These are defined by considering the natural bundles K and L over S_1 (which are, respectively, the kernel and cokernel of df) and looking at the natural map of fiber bundles (over S_1):

$$S_1^{(2)} \to \mathrm{Hom}(K \odot K, L).$$

As before this is a submersion and the range is stratified by the corank r of the corresponding map in $\mathrm{Hom}(K, \mathrm{Hom}(K, L)$. Pulling back the stratification to $S_1^{(2)}$ defines the manifolds $S_{1,r}$.

This allows us to define several new strata as follows:

$$S_{[001],0} := S_{[001]}^{(2)} \cap S_{1,0}$$
$$S_{[111],0} := S_{[111]}^{(2)} \cap S_{1,0}$$
$$S_{[111],1} := S_{[111]}^{(2)} \cap S_{1,1}$$

The first two of these are clearly manifolds, since $S_{1,0}$ is open in $S_1^{(2)}$. The last one is also a manifold, which can be seen by choosing local coordinates. This leaves three intersections which give us a preliminary stratification,

$$S_{[001],1} := S_{[011]}^{(2)} \cap S_{1,1}$$
$$S_{[011],0} := S_{[011]}^{(2)} \cap S_{1,0}$$
$$S_{[011],1} := S_{[011]}^{(2)} \cap S_{1,1}.$$

The first and second of these will be divided into two strata each, and the last into four. Let us first consider the $S_{[001]}$ stratum. Over this stratum there are three important vector bundles K_1, K_2 and L, which are the kernel of df, the kernel of $d(p^2 f)$, and the cokernel of df, respectively. Thus K_2 is 2-dimensional and contains K_1 as a 1-dimensional subspace. The jet space induces a natural bundle map

$$S_{[001]}^{(2)} \to \mathrm{Hom}(T(\Sigma \times I), TJ^1(\Sigma \times I, \mathbb{R}^2 \times I))$$

and so we may project to the normal bundle of $S_{[001]}$ (which, recall, is isomorphic to $\mathrm{Hom}(K, L)$) and restrict to both of the two kernels. Thus, there exist natural natural maps of fiber bundles over $S_{[001]}$,

$$S^{(2)}_{[001]} \to \mathrm{Hom}(K_2 \odot K_1, L) \to \mathrm{Hom}(K_1 \odot K_1, L).$$

We may attempt to stratify $S^{(2)}_{[001]}$ by the coranks of the corresponding operators in the spaces $\mathrm{Hom}(K_2, \mathrm{Hom}(K_1, L)) \cong \mathrm{Hom}(K_1, \mathrm{Hom}(K_2, L))$ and $\mathrm{Hom}(K_1, \mathrm{Hom}(K_1, L))$. The possibilities are labeled $[r\ s]$, where r is the corank of the operator in $\mathrm{Hom}(K_2, \mathrm{Hom}(K_1, L))$ (which agrees with the corank of the operator in $\mathrm{Hom}(K_1, \mathrm{Hom}(K_2, L))$) and s is the corank of the operator in $\mathrm{Hom}(K_1, \mathrm{Hom}(K_1, L))$. Thus there are three possibilities: [00], [01], and [11], corresponding to a stratification of $S^{(2)}_{[001]}$ into three submanifolds $S_{[001],[00]}$, $S_{[001],[01]}$, and $S_{[001],[11]}$. The first of these is exactly the stratum $S_{[001],0}$, which we have already defined. Proving that these are submanifolds of the jet space of the claimed codimensions is now relatively straightforward.

The $S_{[011]}$ stratum leads to more possibilities, as we shall see. Over the stratum $S_{[011]}$, there are several naturally defined bundles. There are three fundamental bundles K_1, K_2, and L, which correspond to the kernel of df, the kernel of $d(pf)$ and the cokernel of df, respectively. $S_{[011]}$ sits inside S_1, and so we may restrict the normal bundle of S_1 to $S_{[011]}$. This is the bundle $\mathrm{Hom}(K_1, L)$. In addition, we may consider the normal bundle of $S_{[011]}$ itself. This is the bundle $\mathrm{Hom}(K_2, L)$. This leads to the following sequence of fiber bundle maps over $S_{[011]}$,

$$S^{(2)}_{[011]} \to \mathrm{Hom}(K_2 \odot K_2, L) \to \mathrm{Hom}(K_2 \odot K_1, L) \to \mathrm{Hom}(K_1 \odot K_1, L).$$

Each of these spaces, $\mathrm{Hom}(K_i \odot K_j, L)$ can be stratified according to the corresponding corank in $\mathrm{Hom}(K_i, \mathrm{Hom}(K_j, L))$. This leads to the following six possibilities:

$$[001],\ [000],\ [100],\ [101],\ [111],\ \text{ and }\ [211]$$

where $[r\ s\ t]$ corresponds to the case where the element in $\mathrm{Hom}(K_2, \mathrm{Hom}(K_2, L))$ has corank r, the element in $\mathrm{Hom}(K_2, \mathrm{Hom}(K_1, L))$ has corank s, and the element in $\mathrm{Hom}(K_1, \mathrm{Hom}(K_1, L))$ has corank t. Again it is straightforward to check that these are submanifolds of the jet space of the claimed codimensions.

The Thom-Boardman stratification continues and our remaining strata may be defined using the Thom-Boardman stratification together with the strata we have already

constructed. The normal bundle to $S_{1,1}$ in $S_1^{(2)}$ is isomorphic to the bundle $\mathrm{Hom}(K \odot K, L)$ and thus, by the same projection-restirction technique we have been employing, there is a natural map of fiber bundles over $S_{1,1}$,

$$S_{1,1}^{(3)} \to \mathrm{Hom}(K \odot K \odot K, L) \to \mathrm{Hom}(K, \mathrm{Hom}(K \odot K, L)).$$

This allows use to stratify $S_{1,1}^{(3)}$ into two strata $S_{1,1,0}$ and $S_{1,1,1}$, based on the corank of the operator in $\mathrm{Hom}(K, \mathrm{Hom}(K \odot K, L))$. Similarly, the normal bundle of $S_{1,1,1}$ is isomorphic to,

$$\mathrm{Hom}(K \odot K \odot K, L)$$

and so we get a map of fiber bundles (over $S_{1,1,1}$),

$$S_{1,1,1}^{(4)} \to \mathrm{Hom}(K \odot K \odot K \odot K, L) \to \mathrm{Hom}(K, \mathrm{Hom}(K \odot K \odot K, L)).$$

Again we can stratify according to the corank of the final operator, yielding two strata: $S_{1,1,1,0}$ and $S_{1,1,1,1}$. The later has codimension four. We can define the remaining strata as the intersections:

$$\begin{aligned}
S_{[001],[01],0} &:= S^{(3)}_{[001],[01]} \cap S_{1,1,0} \\
S_{[001],[01],1} &:= S^{(3)}_{[001],[01]} \cap S_{1,1,1} \\
S_{[001],[11],0} &:= S^{(3)}_{[001],[11]} \cap S_{1,1,0} \\
S_{[001],[11],1} &:= S^{(3)}_{[001],[11]} \cap S_{1,1,1} \\
S_{[011],[001],0} &:= S^{(3)}_{[011],[001]} \cap S_{1,1,0} \\
S_{[011],[001],1} &:= S^{(3)}_{[011],[001]} \cap S_{1,1,1} \\
S_{[011],[100],0} &:= S^{(3)}_{[011],[100]} \cap S_{1,1,0} \\
S_{[011],[100],1} &:= S^{(3)}_{[011],[100]} \cap S_{1,1,1} \\
S_{[001],[01],1,0} &:= S^{(4)}_{[001],[01]} \cap S_{1,1,1,0} \\
S_{[001],[01],1,1} &:= S^{(4)}_{[001],[01]} \cap S_{1,1,1,1}
\end{aligned}$$

Just as the Cerf theory strata were compatible with the Morse theory strata, the 3D strata listed in Table 2.2 are compatible with 2D strata considered in the previous two sections. In particular if $f_0, f_1 : \Sigma^2 \to \mathbb{R}^2$ are generic with respect to the 2D stratification and $F : \Sigma \times I \to \mathbb{R}^2 \times I$ is any map satisfying condition (1) of the Relative Transversality

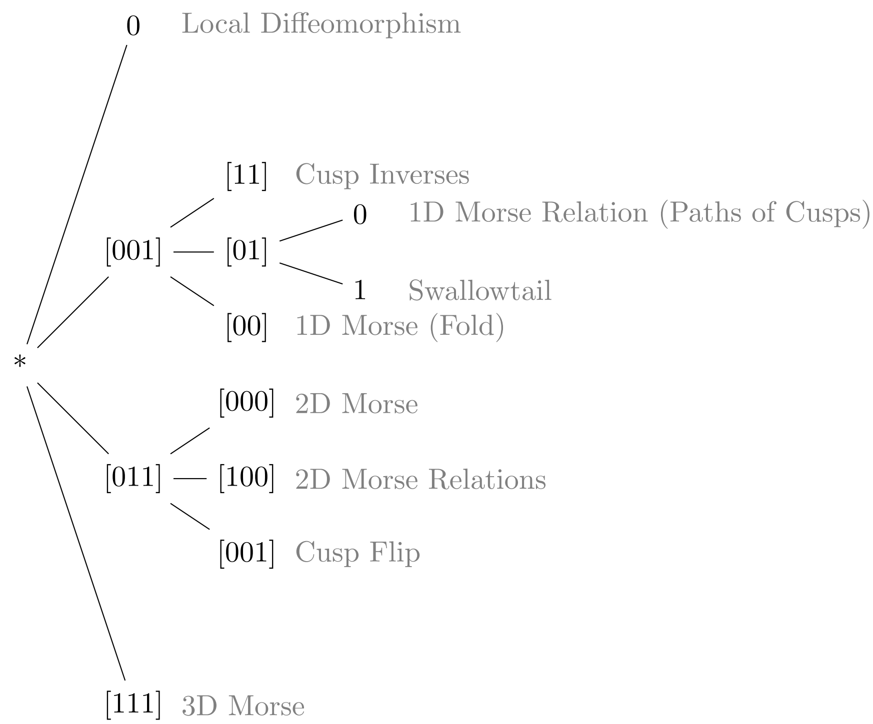


Figure 2.4: The 3D Singularity Family Tree

Theorem 2.1.7, then F automatically satisfies condition (2), as well. Moreover, given such a pair of generic maps, we may choose a path $F : \Sigma \times I \to \mathbb{R}^2 \times I$ satisfying condition (1) of the Theorem 2.1.7. We will be interested in studying those F which are close to being paths. Consider the projection of jet bundles,

$$J^1(\Sigma \times I, \mathbb{R}^2 \times I) \to J^1(\Sigma \times I, I).$$

Inside $J^1(\Sigma \times I, I)$ there is a distinguished open subset $\mathcal{O}$, whose inverse image in $J^1(X \times I, \mathbb{R} \times I)$ is also open (we denote this open set $\mathcal{O}$). $\mathcal{O}$ is defined on considering, on $\Sigma \times I$, the distinguished vector field, which in local coordinates is given by ∂_t, where ∂_t is the standard vector field on I. The span of this gives us 1-dimensional subbundle $E \subseteq T(\Sigma \times I)$, and thus we may consider the projection,

$$J^1(X\Sigma \times I, I) = T^*(\Sigma \times I) \otimes TI \to E^* \otimes TI.$$

The inverse image of the complement of the zero section of $E^* \otimes TI$ gives the open set $\mathcal{O}$.

Thus we may consider the subset of $C^\infty(X \times I, \mathbb{R} \times I)$ given by,

$$M(\mathcal{O}) = \{F \mid j^1 F(\Sigma \times I) \subset \mathcal{O}\}.$$

This is open in the Whitney C^1-topology, and hence open in the Whitney C^∞-topology. It consists of those maps $F(x_0, x_1, t) = (f_t(x_0, x_1), g(x_0, x_1, t))$ such that the t-derivative of the t^{th}-coordinate (i.e. $\partial_t g$) is non-zero at all points of $\Sigma \times I$. A path of functions, $F : \Sigma \times I \to \mathbb{R} \times I$, is a map in $M(\mathcal{O})$, so that we see that $M(\mathcal{O})$ is non-empty. Fix two maps $f_0, f_1 : \Sigma^2 \to \mathbb{R}^2$ which are generic with respect to the 2D stratification. We will restrict our attention to only those maps $F \in M(\mathcal{O})$ which satisfy condition (1) of Theorem 2.1.7.

Definition 2.2.12. Let $f_0, f_1 : \Sigma^2 \to \mathbb{R}^2$ be two maps which are generic with respect to the 2D stratification. A map $F : \Sigma \times I \to \mathbb{R} \times I$ such that $F \in M(\mathcal{O})$ and which satisfies condition (1) of Theorem 2.1.7 will be called *generic* if it lies in the residual (hence dense) subset of those F whose induced jet sections $j^k F$ are transverse to the strata listed in Table 2.2. We will say that the generic map F *connects* the generic maps f_0 and f_1. $\diamond$

2.2.7 Derivation of Local Coordinates: 3D Case

In the last section we have seen that a generic map $F : \Sigma \times I \to \mathbb{R} \times I$ (connecting generic maps $f_0, f_1 : \Sigma^2 \to \mathbb{R}^2$) has singularities of only eight types. Using the Thom-Boardman stratification of jet space, we may determine the local ring of each of these singularities, which are listed in Table 2.3, along with their codimensions. In this section, under the hypothesis that F is generic, we will derive local coordinates around each of these singularities.

The stratification of jet space the we use in the 3-dimensional case considered here was derived by incorporating information about the composition of F with the projections $\mathbb{R}^2 \times I \to \mathbb{R} \times I$ and $\mathbb{R} \times I \to I$. It is natural for us to try to consider those coordinates changes which preserve some of this relevant structure. For all but the S_0 singularity in Table 2.3, there two associated subspace $K_1 \subseteq T_x(\Sigma \times I)$ and $K_2 \subseteq T_x(\Sigma \times I)$, where $x \in \Sigma \times I$ is a critical point of the given singularity type. The coordinate changes that we will consider will preserve these spaces in a sense that we will make precise later.

Our derivation of local coordinates will generally consist of four stages. First we use the implicit function theorem to obtain initial coordinates and we analyze what

Table 2.3: The Local Rings of the 3D Singularities

Singularity Stratum	Name	Codim.	Local Ring
S_0	Local Diffeomorphism	0	$\mathbb{R}$
$S_{[001],0}$	1D Morse (Fold)	1	$\mathbb{R}[x]/(x^2)$
$S_{[011],0}$	2D Morse	2	$\mathbb{R}[x]/(x^2)$
$S_{[011],[100],0}$	2D Morse Relation	3	$\mathbb{R}[x]/(x^2)$
$S_{[001],[11],0}$	Cusp Inversion	3	$\mathbb{R}[x]/(x^3)$
$S_{[001],[01],0}$	1D Morse Relation (Paths of Cusps)	2	$\mathbb{R}[x]/(x^3)$
$S_{[011],[001],0}$	Cusp Flip	3	$\mathbb{R}[x]/(x^3)$
$S_{[001],[01],1,0}$	Swallowtail	3	$\mathbb{R}[x]/(x^4)$

conditions F must satisfy in order that the sections j^kF be transverse to the strata of Table 2.2. Second, using Lemma 2.2.13, we will obtain a new, simplified coordinate system in which the higher partial derivatives of F take convenient values. Next, using the local ring structure induced by F, we will employ the Generalized Malgrange Preparation Theorem 2.2.6 and Corollary 2.2.7 to obtain a functional equation, just as we have done previously in our discussion of Cerf theory. Finally, using this functional equation and the conditions on the partial derivatives of F that we obtained in step two, we will be able to derive a final coordinate system in which F has a simple form. The proofs of the following lemmas is trivial.

Lemma 2.2.13. *Let $h : \mathbb{R}^3 \to \mathbb{R}$ be a function such that $dh = 0$ at the origin. Let (x, y, z) denote the standard coordinates on $\mathbb{R}^3$, and consider the following coordinate change:*

$$u = x + Ay + Bz$$

$$v = y + Cz$$

$$w = z$$

Then the the following partial derivatives of h at the origin with respect to u, v, w, are given

by:

$$
\begin{aligned}
h_{uu} &= h_{xx} & h_{uuw} &= h_{xxz} + (-B + AC)h_{xxx} - Ch_{xxy} \\
h_{uv} &= h_{xy} - Ah_{xx} & h_{uuuv} &= h_{xxxy} - Ah_{xxxx} \\
h_{uw} &= h_{xz} + (-B + AC)h_{xx} - Ch_{xy} & h_{uuuw} &= h_{xxxz} + (-B + AC)h_{xxxx} - Ch_{xxxy} \\
h_{uuv} &= h_{xxy} - Ah_{xxx}
\end{aligned}
$$

where $h_{\alpha\beta}$, $h_{\alpha\beta\gamma}$, and $h_{\alpha\beta\gamma\delta}$ with $\alpha, \beta, \gamma, \delta \in \{x, y, z\}$ denotes the respective partial derivative of h at the origin. Moreover, we have equality of the following tangent vectors at the origin:

$$
\begin{aligned}
\partial_u &= \partial_x \\
\partial_v &= \partial_y - A\partial_x \\
\partial_w &= \partial_z + (-B + AC)\partial_x - C\partial_y
\end{aligned}
$$

Lemma 2.2.14. *Let $h : \mathbb{R}^3 \to \mathbb{R}$ be a function such that $dh = 0$ at the origin. Let (x, y, z) denote the standard coordinates on $\mathbb{R}^3$, and consider the following coordinate change:*

$$
\begin{aligned}
u &= x + Ax^2 \\
v &= y \\
w &= z
\end{aligned}
$$

Then the the following partial derivatives of h at the origin with respect to u, v, w, are given by:

$$
\begin{aligned}
h_{uu} &= h_{xx} - 2Ah_x & h_{uuw} &= h_{xxz} - 2Ah_{xz} \\
h_{uv} &= h_{xy} & h_{uuuv} &= h_{xxxy} - 4Ah_{xxy} + 12h_{xy} \\
h_{uw} &= h_{xz} & h_{uuuw} &= h_{xxxz} - 4Ah_{xxz} + 12h_{xz} \\
h_{uuv} &= h_{xxy} - 2Ah_{xy}
\end{aligned}
$$

where $h_{\alpha\beta}$, $h_{\alpha\beta\gamma}$, and $h_{\alpha\beta\gamma\delta}$ with $\alpha, \beta, \gamma, \delta \in \{x, y, z\}$ denotes the respective partial derivative of h at the origin. Moreover, we have equality of the following tangent vectors:

$$
\begin{aligned}
\partial_u &= \frac{1}{1 + 2Ax}\partial_x \\
\partial_v &= \partial_y \\
\partial_w &= \partial_z
\end{aligned}
$$

The singularities of type S_0 are not actually singularities at all. In a neighborhood of such a point F is a diffeomorphism, and so we call these points *local diffeomorphism points*. In this case F itself provides standard coordinates around such a point. The remaining singularities fall into two kinds, $S_{[001]}$-singularities and $S_{[011]}$-singularities, and our initial coordinates will only depend on this division.

Let (y_0, y_1, y_2) denote the standard coordinates of $\mathbb{R}^2 \times I$ (here y_2 is the standard coordinate of I). Let $x \in \Sigma \times I$ be a point where F obtains one of the seven remaining singularities. The implicit function theorem then ensures that there exist coordinates (x_0, x_1, x_2) of $\Sigma \times I$, centered at x, in which F has the form listed in Table 2.4. In both cases x_2 will coincide with the standard coordinate on I. Moreover in each of these coordinates the function h, defined in Table 2.4, satisfies $dh = 0$.

Table 2.4: Initial Choice of Coordinates

Type	F	dF	K_1	K_2
$S_{[001]}$	$\begin{aligned} F^*y_0 &= h(x_0, x_1, x_2) \\ F^*y_1 &= x_1 \\ F^*y_2 &= x_2 \end{aligned}$	$dF = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\mathrm{Span}\{\partial_{x_0}\}$	$\mathrm{Span}\{\partial_{x_0}, \partial_{x_1}\}$
$S_{[011]}$	$\begin{aligned} F^*y_0 &= x_0 \\ F^*y_1 &= h(x_0, x_1, x_2) \\ F^*y_2 &= x_2 \end{aligned}$	$dF = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\mathrm{Span}\{\partial_{x_1}\}$	$\mathrm{Span}\{\partial_{x_0}, \partial_{x_1}\}$

Recall that over the strata $S_{[001]}$ and $S_{[011]}$ there are vector bundles $K_1 \subset K_2 \subset T(\Sigma \times I)$, defined as the kernel of dF and $d(pF)$, respectively. At the point x these give a pair of subspaces K_1 and K_2 of $T_x(\Sigma \times I)$, which are also listed in Table 2.4. As we saw in the previous section, these subspaces play a key role in the further stratification of the jet space. All the coordinate changes that we will consider will preserve these spaces in the sense that, in the $S_{[001]}$-case, if $(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ are our final coordinates, then we have $K_1 = \mathrm{Span}\{\partial_{\overline{x}_0}\}$ and $K_2 = \mathrm{Span}\{\partial_{\overline{x}_0}, \partial_{\overline{x}_1}\}$ at x. Similarly, in the $S_{[011]}$-case, if $(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ are our final coordinates, then we have $K_1 = \mathrm{Span}\{\partial_{\overline{x}_1}\}$ and $K_2 = \mathrm{Span}\{\partial_{\overline{x}_0}, \partial_{\overline{x}_1}\}$ at x. We

will precede with the codimension 3 singularities first.

Proposition 2.2.15 (Swallowtail)**.** *Let $x \in \Sigma \times I$ be a point where F obtains an $S_{[001],[01],1,0}$ (i.e. Swallowtail) singularity. Then there exist local coordinates $(\overline{x}_0, \overline{x}_1, \overline{x}_2)$ for $\Sigma \times I$ centered at x, and local coordinates $(\overline{y}_0, \overline{y}_1, \overline{y}_2)$ for $\mathbb{R}^2 \times I$ centered at $y = F(x)$, such that in these coordinates F takes the form:*

$$F^*\overline{y}_0 = \pm\overline{x}_0^4 + \overline{x}_1\overline{x}_0 \pm \overline{x}_2\overline{x}_0^2$$
$$F^*\overline{y}_1 = \overline{x}_1$$
$$F^*\overline{y}_2 = \overline{x}_2.$$

Moreover these coordinate changes preserve the orientations of $\mathbb{R}^2 \times I$, $\mathbb{R} \times I$ and I.

Proof. As mentioned earlier, the implicit function theorem guarantees that there exist coordinates with the properties listed in Table 2.4, i.e.

$$F^*y_0 = h(x_0, x_1, x_2)$$
$$F^*y_1 = x_1$$
$$F^*y_2 = x_2.$$

where $dh = 0$ at the origin. Let us expand h in the variables x_0, x_1, x_2. We have,

$$\begin{aligned} h(x_1, x_2, x_3) = &\sum_{0 \le i \le j \le 2} \alpha_{ij} x_i x_j + \sum_{0 \le i \le j \le k \le 2} \beta_{ijk} x_i x_j x_k \\ &+ \sum_{0 \le i \le j \le k \le \ell 2} \gamma_{ijk\ell} x_i x_j x_k x_\ell + O(|x|^5). \end{aligned}$$

Since x is an $S_{[001],[01],1,0}$ singularity, we know that $\alpha_{00} = 0$, $\alpha_{01} \neq 0$, $\beta_{000} = 0$, and $\gamma_{0000} \neq 0$.

By assumption, j^kF is transverse to the strata $S_{[001]}$, $S_{[001],[01]}$, and $S_{[001],[01],1}$ for $k = 1$, 2, and 3, respectively. Thus the following maps are surjective at x:

$$\begin{aligned} d(j^1F) : T_x(\Sigma \times I) \to\ & T_{j^1F(x)} J^1(\Sigma \times I, \mathbb{R}^2 \times I)/T_{j^1F(x)} S_{[001]} \\ &\cong \nu_{j^1F(x)} S_{[001]} \cong \mathrm{Hom}(K_1, L) \\ d(j^2F) : T_x(\Sigma \times I) \to\ & T_{j^2F(x)} J^2(\Sigma \times I, \mathbb{R}^2 \times I)/T_{j^2F(x)} S_{[001],[01]} \\ &\cong \nu_{j^2F(x)} S_{[001],[01]} \cong \mathrm{Hom}(K_1, L) \oplus \mathrm{Hom}(K_1 \cdot K_1, L) \\ d(j^3F) : T_x(\Sigma \times I) \to\ & T_{j^3F(x)} J^3(\Sigma \times I, \mathbb{R}^2 \times I)/T_{j^3F(x)} S_{[001],[01],1} \\ &\cong \nu_{j^3F(x)} S_{[001],[01],1} \\ &\cong \mathrm{Hom}(K_1, L) \oplus \mathrm{Hom}(K_1 \cdot K_1, L) \oplus \mathrm{Hom}(K_1 \cdot K_1 \cdot K_1, L) \end{aligned}$$

Given that $\alpha_{00} = 0$, $\alpha_{01} \neq 0$, $\beta_{000} = 0$, and $\gamma_{0000} \neq 0$, this is equivalent to the condition that the 2×2 matrix,

$$\begin{pmatrix} \alpha_{01} & \alpha_{02} \\ \beta_{001} & \beta_{002} \end{pmatrix}$$

is non-degenerate.

Using an combination of Lemmas 2.2.13 and 2.2.14 we may find coordinates new coordinates (x_0', x_1', x_2') centered at x and (y_0, y_1', y_2) centered at $y = F(x)$, (here y_0 and y_2 are standard coordinates) such that in these coordinates we have,

$$\begin{pmatrix} \alpha_{00} & \alpha_{01} & \alpha_{02} \\ \beta_{000} & \beta_{001} & \beta_{002} \\ \gamma_{0000} & \gamma_{0001} & \gamma_{002} \end{pmatrix} = \begin{pmatrix} 0 & \pm r & 0 \\ 0 & 0 & \pm s \\ \pm t & 0 & 0 \end{pmatrix}. \tag{2.2.16}$$

for some strictly positive constants r, s, t.

The local ring is given by $\mathcal{R}_F = \mathbb{R}[x_0']/((x_0')^4)$ and hence, by the Generalized Malgrange Preparation Theorem 2.2.6, there exist functions a, b, c, d on $\mathbb{R}^2 \times I$ such that,

$$(x_0')^4 = F^*a + F^*b \cdot x_0' + F^*c \cdot (x_0')^2 + F^*d \cdot (x_0')^3.$$

By our conditions 2.2.16 and by collecting terms we see that $\frac{\partial d}{\partial y_1}(0) = 0$ and $\frac{\partial d}{\partial y_2}(0) = 0$. Thus we may replace x_0' by $x_0' - \frac{1}{3}F^*d$, without effecting our conditions 2.2.16. In otherwords, we may assume without loss of generality that $d \equiv 0$.

By using our conditions 2.2.16 and again collecting terms, we see that

$$\begin{array}{lll} \dfrac{\partial a}{\partial y_0}(0) = \pm t & \dfrac{\partial a}{\partial y_1}(0) = 0 & \dfrac{\partial a}{\partial y_2}(0) = 0 \\ \dfrac{\partial b}{\partial y_0}(0) = 0 & \dfrac{\partial b}{\partial y_1}(0) = \pm r & \dfrac{\partial b}{\partial y_2}(0) = 0 \\ \dfrac{\partial b}{\partial y_0}(0) = 0 & \dfrac{\partial c}{\partial y_1}(0) = 0 & \dfrac{\partial c}{\partial y_2}(0) = \pm s \end{array}$$

Thus the following pair of coordinate changes are valid in neighborhoods of x and y:

$$\begin{array}{ll} \overline{x}_0 = \pm x_0 & \overline{y}_0 = a(y_0, y_1, y_2) \\ \overline{x}_1 = F^*b & \overline{y}_1 = b(y_0, y_1, y_2) \\ \overline{x}_2 = F^*c & \overline{y}_2 = c(y_0, y_1, y_2) \end{array}$$

In these coordinates F has the desired form. $\square$

Proposition 2.2.17 (Cusp Inversion)**.** *Let $x \in \Sigma \times I$ be a point where F obtains an $S_{[001],[11],0}$ (i.e. Cusp Inversion) singularity. Then there exist local coordinates $(\overline{x}_0, \overline{x}_1, \overline{x}_2)$ for $\Sigma \times I$ centered at x, and local coordinates $(\overline{y}_0, \overline{y}_1, y_2)$ for $\mathbb{R}^2 \times I$ centered at $y = F(x)$ (here y_2 is the standard coordinate of I), such that in these coordinates F takes the form:*

$$\begin{aligned} F^*\overline{y}_0 &= \overline{x}_0^3 \pm \overline{x}_2\overline{x}_0 \pm \overline{x}_1^2\overline{x}_0 \\ F^*\overline{y}_1 &= \overline{x}_1 \\ F^*y_2 &= \overline{x}_2. \end{aligned}$$

Moreover these coordinate changes preserve the orientations of $\mathbb{R}^2 \times I$, $\mathbb{R} \times I$ and I.

Proof. Again, the implicit function theorem guarantees that there exist coordinates with the properties listed in Table 2.4, i.e.

$$\begin{aligned} F^*y_0 &= h(x_0, x_1, x_2) \\ F^*y_1 &= x_1 \\ F^*y_2 &= x_2. \end{aligned}$$

where $dh = 0$. Again we can expand h in the variables x_0, x_1, x_2 to get,

$$h(x_1, x_2, x_3) = \sum_{0 \leq i \leq j \leq 2} \alpha_{ij} x_i x_j + \sum_{0 \leq i \leq j \leq k \leq 2} \beta_{ijk} x_i x_j x_k + O(|x|^4).$$

Since x is an $S_{[001],[11],0}$-singularity, we know that $\alpha_{00} = 0$, $\alpha_{01} = 0$, and $\beta_{000} \neq 0$.

By assumption, j^kF is transverse to the strata $S_{[001]}$ and $S_{[001],[11]}$ for $k = 1$ and 2, respectively. Thus the following maps are surjective at x:

$$\begin{aligned} d(j^1F) : T_x(\Sigma \times I) &\to T_{j^1F(x)} J^1(\Sigma \times I, \mathbb{R}^2 \times I)/T_{j^1F(x)} S_{[001]} \\ &\cong \nu_{j^1F(x)} S_{[001]} \cong \mathrm{Hom}(K_1, L) \\ d(j^2F) : T_x(\Sigma \times I) &\to T_{j^2F(x)} J^2(\Sigma \times I, \mathbb{R}^2 \times I)/T_{j^2F(x)} S_{[001],[11]} \\ &\cong \nu_{j^2F(x)} S_{[001],[11]} \cong \mathrm{Hom}(K_1, L) \oplus \mathrm{Hom}(K_1 \cdot K_2, L) \end{aligned}$$

Given that $\alpha_{00} = \alpha_{01} = 0$ and $\beta_{000} = 0$, this is equivalent to the condition that $\alpha_{02} \neq 0$ and that the following matrix is non-degenerate:

$$\begin{pmatrix} 0 & 0 & \alpha_{02} \\ \beta_{000} & \beta_{001} & \beta_{002} \\ \beta_{001} & \beta_{011} & \beta_{012} \end{pmatrix}$$

Using Lemma 2.2.13 we may find acceptable coordinates (x_0', x_1', x_2') and (y_0, y_1', y_2) in which the above matrix becomes:

$$\begin{pmatrix} 0 & 0 & \pm r \\ \pm s & 0 & 0 \\ 0 & \pm t & 0 \end{pmatrix}$$

for some positive numbers r, s, t. By scaling x_0 appropriately, we may assume that $s = 1$.

The local ring of this singularity is $\mathcal{R}_F \cong \mathbb{R}[x_0']/((x_0')^3)$ and so by the generalized Malgrange Preparation theorem there exist functions smooth a, b, c on $\mathbb{R}^2 \times I$ such that

$$(x_0')^3 = F^*a + F^*b \cdot x_0' + F^*c \cdot (x_0')^2.$$

By suitably changing the coordinate x_0', we may assume that $c \equiv 0$. Moreover, we have $\frac{\partial a}{\partial y_0}(0) \neq 0$, $\frac{\partial a}{\partial y_1}(0) = 0$, and

$$b(y_0, y_1', y_2) = \pm y_2 + g(y_0, y_1, y_2),$$

where the function g has the form,

$$g(y_0, y_1, y_2) = \pm t \cdot y_1^2 + O(|y|^3).$$

Thus the following is a valid pair of coordinate changes in sufficiently small neighborhoods of x and y:

$$\begin{aligned} \overline{x}_0 &= \pm x_0' & \qquad \overline{y}_0 &= a(y_0, y_1, y_2) \\ \overline{x}_1 &= \sqrt{F^*g} & \overline{y}_1 &= \sqrt{g(y_0, y_1, y_2)} \\ \overline{x}_2 &= x_2' & \overline{y}_2 &= y_2 \end{aligned}$$

In these coordinates F has the desired form. □

Proposition 2.2.18 (Cusp Flip)**.** *Let $x \in \Sigma \times I$ be a point where F obtains an $S_{[011],[001],0}$ (i.e. Cusp Flip) singularity. Then there exist local coordinates $(\overline{x}_0, \overline{x}_1, \overline{x}_2)$ for $\Sigma \times I$ centered at x, and local coordinates $(\overline{y}_0, \overline{y}_1, \overline{y}_2)$ for $\mathbb{R}^2 \times I$ centered at $y = F(x)$, such that in these coordinates F takes the form:*

$$\begin{aligned} F^*\overline{y}_0 &= \overline{x}_0 \\ F^*\overline{y}_1 &= \overline{x}_1^3 \pm \overline{x}_1\overline{x}_0 \pm \overline{x}_1^2\overline{x}_2 \\ F^*\overline{y}_2 &= \overline{x}_2. \end{aligned}$$

Moreover these coordinate changes preserve the orientations of $\mathbb{R}^2 \times I$, $\mathbb{R} \times I$ and I.

Proof. The implicit function theorem implies that there are coordinates (x_0, x_1, x_2) centered at x with the properties listed in Table 2.4, i.e.

$$\begin{aligned} F^* y_0 &= x_0 \\ F^* y_1 &= h(x_0, x_1, x_2) \\ F^* y_2 &= x_2. \end{aligned}$$

where $dh = 0$ at the origin. Expanding h in the variables x_0, x_1, x_2 we get

$$h(x_1, x_2, x_3) = \sum_{0 \le i \le j \le 2} \alpha_{ij} x_i x_j + \sum_{0 \le i \le j \le k \le 2} \beta_{ijk} x_i x_j x_k + O(|x|^4).$$

Since x is an $S_{[011],[001],0}$-singularity we know that $\alpha_{01} \neq 0$, $\alpha_{11} = 0$, and $\beta_{111} \neq 0$.

By assumption, $j^k F$ is transverse to the strata $S_{[011]}$ and $S_{[011],[001]}$ for $k = 1$, and 2, respectively. Thus the following maps are surjective at x:

$$\begin{aligned} d(j^1 F) : T_x(\Sigma \times I) &\to T_{j^1 F(x)} J^1(\Sigma \times I, \mathbb{R}^2 \times I) / T_{j^1 F(x)} S_{[011]} \\ &\cong \nu_{j^1 F(x)} S_{[011]} \cong \mathrm{Hom}(K_2, L) \\ d(j^2 F) : T_x(\Sigma \times I) &\to T_{j^2 F(x)} J^2(\Sigma \times I, \mathbb{R}^2 \times I) / T_{j^2 F(x)} S_{[011],[001]} \\ &\cong \nu_{j^2 F(x)} S_{[011],[001]} \cong \mathrm{Hom}(K_2, L) \oplus \mathrm{Hom}(K_1 \cdot K_1, L) \end{aligned}$$

Given that $\alpha_{11} = 0$, $\alpha_{01} \neq 0$, and $\beta_{111} \neq 0$, this is equivalent to the condition that the 3×3 matrix,

$$\begin{pmatrix} \alpha_{01} & 0 & \alpha_{12} \\ \alpha_{00} & \alpha_{01} & \alpha_{02} \\ \beta_{011} & \beta_{111} & \beta_{112} \end{pmatrix}$$

is non-degenerate. Using Lemma 2.2.13 and Lemma 2.2.14 we may find acceptable coordinates (x_0', x_1', x_2') and (y_0', y_1', y_2') in which the above matrix becomes:

$$\begin{pmatrix} \pm r & 0 & 0 \\ 0 & \pm r & 0 \\ 0 & \pm s & \pm t \end{pmatrix}$$

for some positive numbers r, s, t.

The local ring of this singularity is $\mathcal{R}_F \cong \mathbb{R}[x_1']((x_1')^3)$, and so by the Malgrange Preparation Theorem there exist functions a, b, c on $\mathbb{R}^2 \times I$ such that,

$$(x_1')^3 = F^* a + F^* b \cdot x_1' + F^* c \cdot (x_1')^2.$$

Collecting terms we see that,

$$\begin{array}{lll}
\frac{\partial a}{\partial y_0}(0) = 0 & \frac{\partial a}{\partial y_1}(0) \neq 0 & \frac{\partial a}{\partial y_2}(0) = 0 \\
\frac{\partial b}{\partial y_0}(0) \neq 0 & \frac{\partial b}{\partial y_1}(0) = 0 & \frac{\partial b}{\partial y_2}(0) = 0 \\
\frac{\partial b}{\partial y_0}(0) = 0 & \frac{\partial c}{\partial y_1}(0) = 0 & \frac{\partial c}{\partial y_2}(0) \neq 0
\end{array}$$

Thus the following pair of coordinate changes are valid in neighborhoods of x and y:

$$\begin{array}{ll}
\overline{x}_0 = F^*b & \overline{y}_0 = b(y_0', y_1', y_2') \\
\overline{x}_1 = \pm x_1' & \overline{y}_1 = a(y_0', y_1', y_2') \\
\overline{x}_2 = F^*c & \overline{y}_2 = c(y_0', y_1', y_2')
\end{array}$$

In these coordinates F has the desired form. $\square$

Proposition 2.2.19 (2D Morse Relation)**.** *Let $x \in \Sigma \times I$ be a point where F obtains an $S_{[011],[100],0}$ (i.e. 2D Morse Relation) singularity. Then there exist local coordinates $(\overline{x}_0, \overline{x}_1, \overline{x}_2)$ for $\Sigma \times I$ centered at x, and local coordinates $(\overline{y}_0, \overline{y}_1, \overline{y}_2)$ for $\mathbb{R}^2 \times I$ centered at $y = F(x)$, such that in these coordinates F takes the form:*

$$\begin{aligned}
F^*\overline{y}_0 &= \overline{x}_0 \\
F^*\overline{y}_1 &= \pm\overline{x}_1^2 \pm \overline{x}_0\overline{x}_2 \pm \overline{x}_0^3 \\
F^*\overline{y}_2 &= \overline{x}_2.
\end{aligned}$$

Moreover these coordinate changes preserve the orientations of $\mathbb{R}^2 \times I$, $\mathbb{R} \times I$ and I.

Proof. The implicit function theorem implies that there are coordinates (x_0, x_1, x_2) centered at x with the properties listed in Table 2.4, i.e.

$$\begin{aligned}
F^*y_0 &= x_0 \\
F^*y_1 &= h(x_0, x_1, x_2) \\
F^*y_2 &= x_2.
\end{aligned}$$

where $dh = 0$ at the origin. Expanding h in the variables x_0, x_1, x_2 we get

$$h(x_1, x_2, x_3) = \sum_{0 \leq i \leq j \leq 2} \alpha_{ij} x_i x_j + \sum_{0 \leq i \leq j \leq k \leq 2} \beta_{ijk} x_i x_j x_k + O(|x|^4).$$

Since x is an $S_{[011],[100],0}$-singularity we know that $\alpha_{11} \neq 0$, and that

$$\alpha_{01}^2 - \alpha_{11}\alpha_{00} = 0.$$

By assumption, $j^k F$ is transverse to the strata $S_{[011]}$ and $S_{[011],[100]}$ for $k = 1$, and 2, respectively. Thus the following maps are surjective at x:

$$\begin{aligned} d(j^1F) : T_x(\Sigma \times I) &\to T_{j^1F(x)} J^1(\Sigma \times I, \mathbb{R}^2 \times I)/T_{j^1F(x)} S_{[011]} \\ &\cong \nu_{j^1F(x)} S_{[011]} \cong \mathrm{Hom}(K_2, L) \\ d(j^2F) : T_x(\Sigma \times I) &\to T_{j^2F(x)} J^2(\Sigma \times I, \mathbb{R}^2 \times I)/T_{j^2F(x)} S_{[011],[100]} \cong \nu_{j^2F(x)} S_{[011],[001]} \end{aligned}$$

This is equivalent to the conditions that the 2×3 matrix,

$$\begin{pmatrix} \alpha_{01} & \alpha_{11} & \alpha_{12} \\ \alpha_{00} & \alpha_{01} & \alpha_{02} \end{pmatrix}$$

has rank two and that the 3×3-matrix,

$$\begin{pmatrix} \alpha_{01} & \alpha_{11} & \alpha_{12} \\ \alpha_{00} & \alpha_{01} & \alpha_{02} \\ & \nabla(\alpha_{01}^2 - \alpha_{11}\alpha_{00}) & \end{pmatrix}$$

is non-degenerate. Using Lemma 2.2.13 and Lemma 2.2.14 we may find acceptable coordinates (x_0', x_1', x_2') and (y_0', y_1', y_2') in which the following equation holds

$$\begin{pmatrix} \alpha_{01} & \alpha_{11} & \alpha_{12} \\ \alpha_{00} & \alpha_{01} & \alpha_{02} \\ \beta_{000} & \beta_{001} & \beta_{002} \end{pmatrix} = \begin{pmatrix} 0 & \pm r & 0 \\ 0 & 0 & \pm s \\ \pm t & 0 & 0 \end{pmatrix}.$$

for some strictly positive constants r, s, t.

The local ring of this singularity is $\mathcal{R}_F \cong \mathbb{R}[x_1']((x_1')^2)$, and so by the Malgrange Preparation Theorem there exist functions a, b, c on $\mathbb{R}^2 \times I$ such that,

$$(x_1')^2 = F^*a + F^*b \cdot x_1',$$

and making a suitable change to the x_1' coordinate we may assume that $b \equiv 0$. Expanding both sides we see that,

$$a(y_0', y_1', y_2') = \tilde{a}(y_0', y_1', y_2') \pm s \cdot y_0' y_2' \pm t \cdot y_0^3,$$

and moreover,

$$\begin{aligned}
\frac{\partial^3 \tilde{a}}{\partial y_0^3}(0) &= 0 \\
\frac{\partial^2 \tilde{a}}{\partial y_0 \partial y_2}(0) &= 0 \\
\frac{\partial^3 \tilde{a}}{\partial y_0^3}(0) &= \pm\frac{1}{t} y_1 + O(|y|^2).
\end{aligned}$$

Thus the following pair of coordinate changes are valid in neighborhoods of x and y:

$$\begin{aligned}
\overline{x}_0 &= x'_0 \sqrt[3]{t} & \overline{y}_0 &= y'_0 \sqrt[3]{t} \\
\overline{x}_1 &= x'_1 & \overline{y}_1 &= \tilde{a}(y'_0, y'_1, y'_2) \\
\overline{x}_2 &= x'_2 \cdot s\frac{1}{\sqrt[3]{t}} & \overline{y}_2 &= y'_2 \cdot s\frac{1}{\sqrt[3]{t}}
\end{aligned}$$

In these coordinates F has the desired form. $\square$

We now turn to the lower codimension singularities. Let $S_{[001],[01],0}(F)$, $S_{[011],0}(F)$, and $S_{[001],0}(F)$ denote those points in $\Sigma^2 \times I$ in which F obtains an $S_{[001],[01],0}$-, $S_{[011],0}$-, or $S_{[001],0}$-singularity, respectively. By transversality, each of these is a smooth submanifold of $\Sigma^2 \times I$ of dimension one, one, and two, respectively. Moreover, in the $S_{[001],[01],0}$ and $S_{[011],0}$ cases, the projections

$$\begin{aligned}
F :& S_{[001],[01],0}(F) \to \mathbb{R}^2 \times I \xrightarrow{p^2} I \\
F :& S_{[011],0}(F) \to \mathbb{R}^2 \times I \xrightarrow{p^2} I,
\end{aligned}$$

are local diffeomorphisms. Thus in both of these cases the image of this submanifold under F is (locally) the image of the graph of some function $\gamma = (\gamma_0, \gamma_1) : I \to \mathbb{R}^2$. Similarly, in the $S_{[001],0}$ case the projection,

$$F : S_{[001],0}(F) \to \mathbb{R}^2 \times I \xrightarrow{p} \mathbb{R} \times I$$

is a local diffeomorphism and hence the image under F of this submanifold is (locally) the graph of a function $\gamma : \mathbb{R} \times I \to \mathbb{R}$. The $S_{[001],0}$ case is the easiest of the three and we begin there.

Proposition 2.2.20 (1D Morse (Fold))**.** *Let $F : \Sigma^2 \times I \to \mathbb{R}^2 \times I$ be generic and let $x \in \Sigma \times I$ be an $S_{[001],0}$ singularity point. Then there exists a submanifold $Y^2 \subset \Sigma \times I$ containing x*

and consisting entirely of $S_{[001],0}$ singularities, a neighborhood $U \subset \mathbb{R} \times I$ of $p(y) = pF(x)$, and a function $\gamma : U \to \mathbb{R}$ whose graph $\{(\gamma(y_1, y_2), y_1, y_2) \mid (y_1, y_2) \in U\}$ consists precisely of the image of Y under F. Moreover there exist coordinates $(\overline{x}_0, \overline{x}_1, x_2)$ for $\Sigma^2 \times I$ centered at x, (here x_2 is the standard coordinate for I) such that in these coordinates $Y = \{\overline{x}_0 = 0\}$ (the $\overline{x}_1$-x_2-plane) and F has the following normal form:

$$\begin{aligned} F^* y_0 &= \pm \overline{x_0}^2 + \gamma(\overline{x}_1, \overline{x}_2) \\ F^* y_1 &= \overline{x}_1 \\ F^* y_2 &= x_2 \end{aligned}$$

Here (y_0, y_1, y_2) are the standard coordinates for $\mathbb{R}^2 \times I$ centered at $y = F(x)$.

Proof. Again, the implicit function theorem guarantees that there exist coordinates around x with the properties listed in Table 2.4, i.e.

$$\begin{aligned} F^* y_0 &= h(x_0, x_1, x_2) \\ F^* y_1 &= x_1 \\ F^* y_2 &= x_2. \end{aligned}$$

where $dh = 0$ along Y. We may arrange for $Y = \{x_0 = 0\}$. Since this is an $S_{[001],0}$ singularity, we know that $h_{x_0 x_0} \neq 0$ along Y, and moreover F satisfies the transversality assumption that $j^1 F$ is transverse to the strata $S_{[001]}$, i.e. the following map is surjective along Y:

$$\begin{aligned} d(j^1 F) : T_x(\Sigma \times I) \to T_{j^1 F(x)} J^1(\Sigma \times I, \mathbb{R}^2 \times I) / T_{j^1 F(x)} S_{[001]} \\ \cong \nu_{j^1 F(x)} S_{[001]} \cong \mathrm{Hom}(K_1, L). \end{aligned}$$

We introduce a new function,

$$g(x_0, x_1, x_2) := h(x_0, x_1, x_2) - \gamma(x_1, x_2).$$

Our assumptions on h guarantee that

$$g(0, x_1, x_2) \equiv 0 \qquad \frac{\partial g}{\partial x_0}(0, x_1, x_2) \equiv 0 \qquad \frac{\partial^2 g}{\partial x_0^2}(0, x_1, x_2) \neq 0$$

and thus we have:

$$h(x_0, x_1, x_2) = x_0^2 \tilde{g}(x_0, x_1, x_2) + \gamma(x_1, x_2).$$

for some function $\tilde{g}$, such that $\tilde{g}(0, x_1, x_2) \neq 0$. Thus in a possibly smaller neighborhood of Y, the following is a valid coordinate change:

$$\begin{aligned}
\overline{x}_0 &= x\sqrt{|\tilde{g}(x_0, x_1, x_2)|} \\
\overline{x}_1 &= x_1 \\
\overline{x}_2 &= x_2.
\end{aligned}$$

In these coordinates F has the desired normal form. □

Proposition 2.2.21 (1D Morse Relation (Paths of Cusps))**.** *Let $F : \Sigma^2 \times I \to \mathbb{R}^2 \times I$ be generic and let $x \in \Sigma \times I$ be an $S_{[001],[01],0}$ singularity point. Then there exists a submanifold $Y^1 \subset \Sigma \times I$ containing x and consisting entirely of $S_{[001],[01],0}$ singularities, a neighborhood $U \subset I$ of $p^2(y) = p^2F(x)$, and a function $\gamma = (\gamma_0, \gamma_1) : U \to \mathbb{R}^2$ whose graph $\{(\gamma_0(y_2), \gamma_1(y_2), y_2) \mid y_2 \in U\}$ consists precisely of the image of Y under F. Moreover there exist coordinates $(\overline{x}_0, \overline{x}_1, x_2)$ for $\Sigma^2 \times I$ centered at x, (here x_2 is the standard coordinate for I) and coordinates $(\overline{y}_0, \overline{y}_1, y_2)$ for $\mathbb{R}^2 \times I$, centered at y (here y_2 is the standard coordinate for I) and such that in these coordinates $Y = \{x_0 = 0, x_1 = 0\}$ (the x_2-axis) and F has the following normal form:*

$$\begin{aligned}
F^* y_0 &= \overline{x_0}^3 \pm \overline{x}_0 \overline{x}_1 + \gamma_0(x_2) \\
F^* y_1 &= x_1 + \gamma_1(x_2) \\
F^* y_2 &= x_2
\end{aligned}$$

These coordinate changes can be taken to be compatible with the orientations of $\mathbb{R}^2 \times I$, $\mathbb{R} \times I$ and I.

Proof. Again, the implicit function theorem guarantees that there exist coordinates around x with the properties listed in Table 2.4, i.e.

$$\begin{aligned}
F^* y_0 &= h(x_0, x_1, x_2) \\
F^* y_1 &= x_1 \\
F^* y_2 &= x_2.
\end{aligned}$$

where $dh = 0$ along Y. Moreover, we may arrange so that in these coordinates $Y = \{(0, \gamma_1(x_2), x_2)\}$. Since this is an $S_{[001],[01],0}$ singularity, we have $h_{x_0x_0} = 0$, $h_{x_0x_1} \neq 0$, and $h_{x_0x_0x_0} \neq 0$ along Y. Next, we preform the following coordinate change:

$$\begin{aligned} x_0' &= x_0 & y_0' &= y_0 - \gamma_0(y_2) \\ x_1' &= x_1 - \gamma_1(x_2) & y_1' &= y_1 - \gamma_1(y_2) \\ x_2' &= x_2 & y_2' &= y_2 \end{aligned}$$

In these coordinates, Y consists of the x_2-axis and its image under F consists of the y_2-axis. We introduce the following new function,

$$g(x_0', x_1', x_2') := h(x_0', x_1' + \gamma_1(x_2'), x_2') - \gamma_0(x_2').$$

Notice that $g = F^*y_0'$. Our assumptions on h guarantee that

$$g(0,0,x_2') \equiv 0 \qquad \frac{\partial g}{\partial x_0'}(0,0,x_2') \equiv 0 \qquad \frac{\partial^2 g}{\partial (x_0')^2}(0,0,x_2') \equiv 0$$
$$\frac{\partial^2 g}{\partial x_0' \partial x_1'}(0,0,x_2') \neq 0 \qquad \frac{\partial^3 g}{\partial (x_0')^3}(0,0,x_2') \neq 0$$

The local ring of this singularity is $\mathbb{R}[x_0']/((x_0')^3)$ and so by the Malgrange Preparation Theorem, there exist smooth functions a, b, c on $\mathbb{R}^2 \times I$ such that

$$(x_0')^3 = F^*a + F^*b \cdot x_0' + F^*c \cdot (x_0')^2.$$

We may rewrite this as

$$(x_0 - \frac{1}{3}F^*c)^3 + F^*\tilde{b} \cdot (x_0 - \frac{1}{3}F^*c) = F^*\tilde{a}$$

for some new functions $\tilde{a}, \tilde{b}$ on $\mathbb{R}^2 \times I$. Collecting terms we see that,

$$\frac{\partial \tilde{a}}{\partial y_0}(0,0,y_2) \neq 0 \qquad \frac{\partial \tilde{b}}{\partial y_0}(0,0,y_2) = 0 \qquad \frac{\partial \tilde{b}}{\partial y_1}(0,0,y_2) \neq 0.$$

Thus the following is a valid pair of coordinate changes:

$$\begin{aligned} \overline{x}_0 &= \pm(x_0' - \frac{1}{3}F^*c) & \overline{y}_0 &= \tilde{a}(y_0' + \gamma(y_2'), y_1', y_2') + \gamma(y_2') \\ \overline{x}_1 &= F^*\tilde{b} & \overline{y}_1 &= \tilde{b}(y_0', y_1', y_2') \\ \overline{x}_2 &= x_2' & \overline{y}_2 &= y_2' \end{aligned}$$

Notice that $\overline{y}_2 = y_2' = y_2$. In these coordinates F has the desired form. □

Proposition 2.2.22 (2D Morse). *Let $F : \Sigma^2 \times I \to \mathbb{R}^2 \times I$ be generic and let $x \in \Sigma \times I$ be an $S_{[011],[000]}$ singularity point. Then there exists a submanifold $Y^1 \subset \Sigma \times I$ containing x and consisting entirely of $S_{[011],[000]}$ singularities, a neighborhood $U \subset I$ of $p^2(y) = p^2 F(x)$, and a function $\gamma = (\gamma_0, \gamma_1) : U \to \mathbb{R}^2$ whose graph $\{(\gamma_0(y_2), \gamma_1(y_2), y_2) \mid y_2 \in U\}$ consists precisely of the image of Y under F. Moreover there exist coordinates $(\overline{x}_0, \overline{x}_1, x_2)$ for $\Sigma^2 \times I$ centered at x, (here x_2 is the standard coordinate for I) and coordinates $(\overline{y}_0, \overline{y}_1, y_2)$ for $\mathbb{R}^2 \times I$, centered at y (here y_2 is the standard coordinate for I) and such that in these coordinates $Y = \{x_0 = 0, x_1 = 0\}$ (the x_2-axis) and F has the following normal form:*

$$
\begin{aligned}
F^* y_0 &= x_0 + \gamma_0(x_2) \\
F^* y_1 &= \pm x_0^2 \pm x_1^2 + \gamma_1(x_2) \\
F^* y_2 &= x_2
\end{aligned}
$$

These coordinate changes can be taken to be compatible with the orientations of $\mathbb{R}^2 \times I$, $\mathbb{R} \times I$ and I.

Proof. The implicit function theorem guarantees that there exist coordinates around x with the properties listed in Table 2.4, i.e.

$$
\begin{aligned}
F^* y_0 &= x_0 \\
F^* y_1 &= h(x_0, x_1, x_2) \\
F^* y_2 &= x_2.
\end{aligned}
$$

where $Y = \{(\gamma_0(x_2), 0, x_2)\}$. Since this is an $S_{[011],[000]}$ singularity, along Y we have $dh \equiv 0$, $\frac{\partial^2 h}{\partial x_1^2} \neq 0$ and

$$
\begin{pmatrix}
\frac{\partial^2 h}{\partial x_0 \partial x_1} & \frac{\partial^2 h}{\partial x_1^2} \\
\frac{\partial^2 h}{\partial x_0^2} & \frac{\partial^2 h}{\partial x_0 \partial x_1}
\end{pmatrix}
$$

is non-singular. Next, we preform the following coordinate change:

$$
\begin{aligned}
x_0' &= x_0 - \gamma_0(x_2) & \qquad y_0' &= y_0 - \gamma_0(y_2) \\
x_1' &= x_1 & \qquad y_1' &= y_1 - \gamma_1(y_2) \\
x_2' &= x_2 & \qquad y_2' &= y_2
\end{aligned}
$$

In these coordinates, Y consists of the x_2-axis and its image consists of the y_2-axis. The define $g = F^* y_1'$. We have,

$$F^* y_1' = g(x_0', x_1', x_2') = h(x_0' + \gamma_0(x_2'), x_1', x_2') - \gamma_1(x_2').$$

Our conditions on F assure

$$g(0,0,x_2') \equiv 0 \qquad\qquad dg(0,0,x_2) \equiv 0$$

$$\begin{pmatrix} \frac{\partial^2 g}{\partial x_0' \partial x_1'} & \frac{\partial^2 g}{\partial (x_1')^2} \\ \frac{\partial^2 g}{\partial (x_0')^2} & \frac{\partial^2 g}{\partial x_0' \partial x_1'} \end{pmatrix} (0,0,x_2') \text{ is non-singular} \qquad\qquad \frac{\partial^2 g}{\partial (x_1')^2}(0,0,x_2) \neq 0.$$

The local ring of this singularity is given by $\mathcal{R}_F = \mathbb{R}[x_1']/((x_1')^2)$ and so by the Malgrange Preparation Theorem there exist functions a, b on $\mathbb{R}^2 \times I$ such that

$$(x_1')^2 = F^* a + F^* b \cdot x_1'.$$

By setting $x_1'' = x_1' - \frac{1}{2} F^* b$, we may assume that $b \equiv 0$. by our conditions on$F$ (and hence g) imply that,

$$a(y_0', y_1', y_2') = A \cdot (y_0')^2 + \tilde{a}(y_0', y_1', y_2')$$

for some non-zero constant A and where,

$$\frac{\partial a}{\partial y_0'}(0,0,y_2') = 0 \qquad\qquad \frac{\partial a}{\partial y_1'}(0,0,y_2') \neq 0$$

$$\frac{\partial a}{\partial y_2'}(0,0,y_2') = 0 \qquad\qquad \frac{\partial^2 a}{\partial (y_0')^2}(0,0,y_2') \neq 0$$

Thus the following is a valid pair of coordinate changes:

$$\begin{aligned} \overline{x}_0 &= \sqrt{|A|} x_0' & \qquad \overline{y}_0 &= \sqrt{|A|} y_0 + \gamma_0(y_2') \\ \overline{x}_1 &= x_1'' & \overline{y}_1 &= \tilde{a}(y_0', y_1', y_2') + \gamma_1(y_2) \\ \overline{x}_2 &= x_2' & \overline{y}_2 &= y_2' \end{aligned}$$

In these coordinates, F has the desired form. □

Remark 2.2.23. The Coordinates for singularities that we have derived often occur in multiple forms. For example there are two forms of the fold singularities of Proposition 2.2.20, according to the sign of x_0-coordinate. Viewing a fold singularity as a path of 1-manifold Morse singularities (which we elaborate on in the following section), we see that these two forms correspond to the two possible indices of such Morse functions. In general we will call these various forms of singularities the *indices* of the singularity. Table 2.5 gives the number of indices in the 3-dimensional case. ⋄

Table 2.5: The Number of Indices of the 3D Singularities

Singularity Stratum	Name	Number of Indices
S_0	Local Diffeomorphism	0
$S_{[001],0}$	1D Morse (Fold)	2
$S_{[011],0}$	2D Morse	4
$S_{[011],[100],0}$	2D Morse Relation	8
$S_{[001],[11],0}$	Cusp Inversion	4
$S_{[001],[01],0}$	1D Morse Relation (Paths of Cusps)	2
$S_{[011],[001],0}$	Cusp Flip	4
$S_{[001],[01],1,0}$	Swallowtail	4

2.3 The Geometry of the Singularities

In the previous sections we derived normal coordinates for each of the types of singularities we encounter in generic maps $\Sigma^2 \to \mathbb{R}^2$ and $\Sigma^2 \times I \to \mathbb{R}^2 \times I$. These normal coordinates allow us to understand the local geometric nature of such generic maps. In this section we explain qualitatively what each of these singularities looks like and how this determines the local structure of Σ.

In Morse theory, one of the pieces of data that one can recover from a Morse function is a picture in the target $\mathbb{R}$ of the singularities. In this 1-dimensional setting, this is not much data, just a finite sequence of isolated critical values. In the higher dimensional setting we are concerned with here, the image in the target of the singular locus carries far more information. Most of the singularities are non-isolated, for example a cusp singularity $S_{[01],1,0}$ occurs at a single point, but it is part of a whole arc of $S_{[01]}$ singularities. The configurations of these singularities amongst each other provides a great deal of local information about the source manifold. Moreover, their image in the target provides a partial picture of the global structure of this manifold, and in subsequent sections this will form the basis of a more elaborate diagram, from which we can recover the source manifold completely.

Given the importance of the image of the singular locus, it is not surprising that it has a name. We call this the *graphic* of the generic map. We will see many examples in what follows. Let us look at our first examples. The fold singularities of Proposition 2.2.11 and their corresponding graphics are depicted in Figure 2.5. To the left of each diagram in the

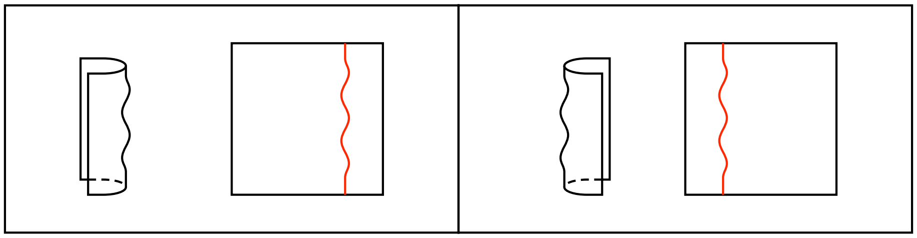


Figure 2.5: Fold Singularities and Their Graphics

figure we have depicted a 3-dimensional embedded surface. The projection of this surface into the plane of the page is a map to $\mathbb{R}^2$ which is generic and exhibits $S_{[01],0}$-singularities.

Notice that these singularities are not isolated and that in the coordinates of Proposition 2.2.11 the singular locus is parametrized by a curve, γ. The graphic of this singularity consists of a smoothly embedded arc in $\mathbb{R}^2$, whose projection to $\mathbb{R}$ is a diffeomorphism. In Figure 2.5 we have depicted this as the wiggly red line. This is how we will typically draw our graphics. The fold singularity can be considered as a vertical path of Morse functions for a 1-manifold.

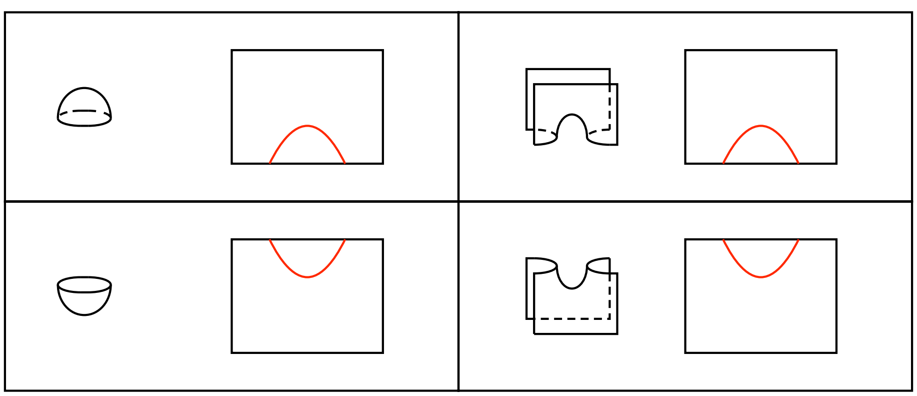


Figure 2.6: 2D Morse Singularities and Their Graphics

The four 2D Morse $(S_{[11]}, 0)$ Singularities of Proposition 2.2.10 are depicted in Figure 2.6 while the two Cusp $(S_{[01],1,0})$ Singularities of Proposition 2.2.9 are depicted in Figure 2.7. Our naming convention should now be clear. The fold singularities are so named because the correspond to singularities which occur when one sheet of the surface Σ is "folded" over another.

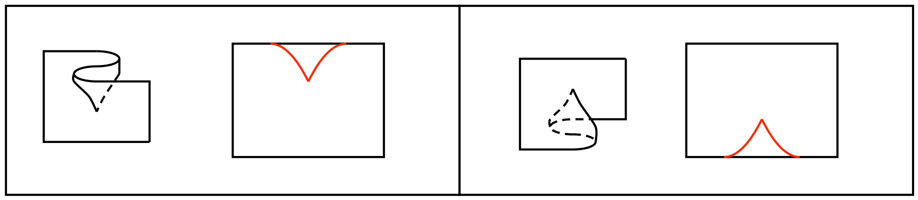


Figure 2.7: Cusp Singularities and Their Graphics

The 2D Morse singularities are similar to the fold singularities, but are distinguished by their projection to $\mathbb{R}$. For the folds, the projection to $\mathbb{R}$ has no critical points, while for the 2D Morse, there is a single isolated Morse singularity. If we were just concerned with the Morse function given by projecting to $\mathbb{R}$, then we would not be able to distinguish between the two "saddles". These would both correspond to Morse singularities of index one. However we don't just have a map to $\mathbb{R}$, we have a map to $\mathbb{R}^2$, and this additional coordinate allows us to distinguish these singularities.

The Cusp singularities are equally well known. It is a singularity that arises in the study of Morse functions via Cerf theory. Our definition of the $S_{[01],1,0}$ stratum ensures that around these singularities the projection to $\mathbb{R}$ can be viewed as a time parameter and that the map to $\mathbb{R}^2$ can be viewed as a path of Morse functions on a 1-dimensional manifold. This path consists entirely of Morse functions except at the Cusp singularity itself, where a birth/death is occurring.

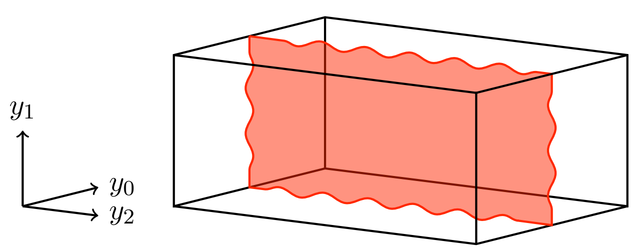


Figure 2.8: Fold Singularities in Three Dimensions

This brings us to the 3-dimensional singularities. These singularities can be divided into those of codimension three and those of lower codimension. The lower codimensional singularities correspond precisely to paths of the previous 2-dimensional singularities. Thus the $S_{[001],0}$-singularities correspond to paths of the fold singularities. The image in $\mathbb{R}^2 \times I$

consists of a 2-dimensional sheet which is swept out, and whose projection to $\mathbb{R} \times I$ is a diffeomorphism, see Figure 2.8.

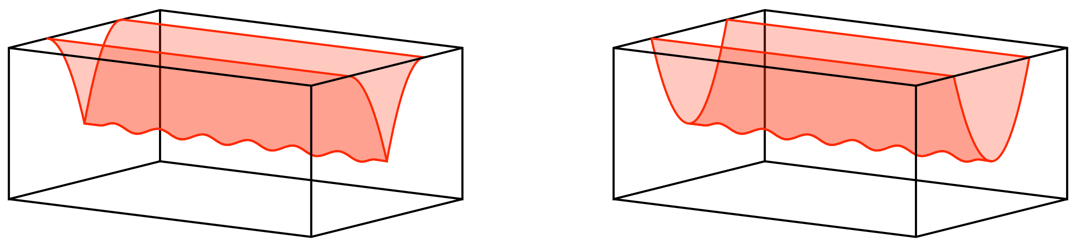


Figure 2.9: Cusp and 2D Morse Singularities in Three Dimensions

Similarly the Cusp singularities $S_{[001],[01],0}$ and the 2D Morse singularities $S_{[011],0}$ have graphics which consist of paths of the previous 2-dimensional graphics. Some of these are depicted in Figure 2.9, with the others obtained by the obvious permutations of these graphics.

The codimension three singularities are more interesting. For example the 2D Morse Relation singularity, $S_{[011],[100],0}$, occurs, as the name suggests, around a path of maps to $\mathbb{R}^2$ in which two 2D Morse singularities form a birth/death. There are a total of eight possibilities for this singularity, and Figure 2.10 shows one of these. This figure shows the initial and final surfaces, together with their 2-dimensional graphics. In the middle pane it shows the full 3-dimensional graphic. The remaining seven versions of this singularity are obtained as the seven obvious permutations of this diagram.

The Cusp Inversion singularities, $S_{[001],[11],0}$, occur in four types, of which two are depicted in Figure 2.11. Again we have show the initial and final surfaces, their 2-dimensional graphics, and the 3-dimensional graphic of the singularity. The remaining two forms of this singularity are given by permuting these. These singularities are important for bordism bicategory because, as we shall see, they witness the fact that the cusp bordisms are inverse to one another.

Perhaps the most interesting of the singularities are the Cusp Flip singularities, $S_{[011],[001],0}$. They involve a non-trivial interaction between Cusp and 2D Morse singularities. They appear in four types, one of which is depicted in Figure 2.12. The remaining types are again given by permutation. In terms of the graphic, they are easy enough to visualize; the point of the cusp simply "flips over," producing a new graphic. We will have more to

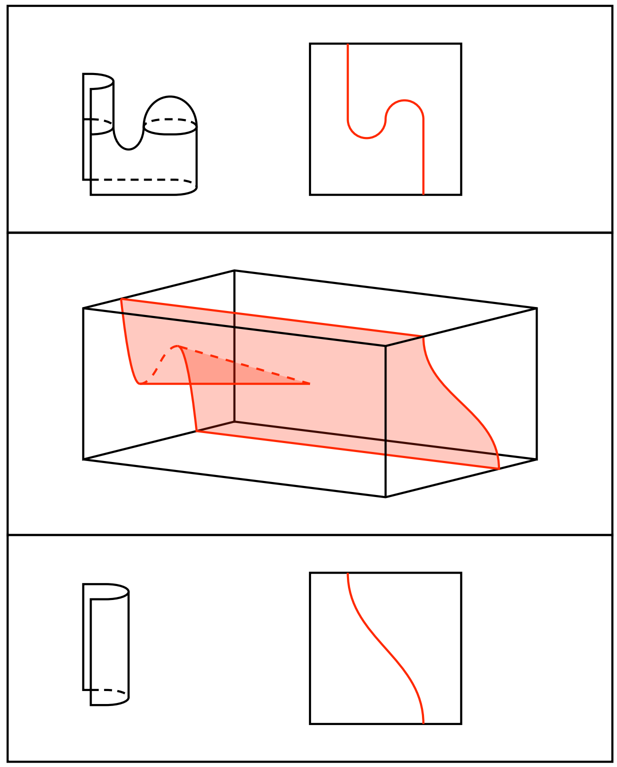


Figure 2.10: 2D Morse Relations and Their Graphics

say about this singularity and its role in the bordism bicategory in Chapter 4.

The final singularity, the Swallowtail Singularity, $S_{[001],[01],1,0}$, is a standard singularity arising from the two parameter version of Cerf theory. One of the four types is depicted in Figure 2.13, and the remaining are given by permuting this diagram. We invite the reader to spend some time visualizing this singularity.

2.4 Multijet Considerations

In the previous sections we have analyzed the local behavior of generic maps $\Sigma \to \mathbb{R}^2$ and $\Sigma \times I \to \mathbb{R}^2 \times I$ and we found that a convenient tool for understanding the structure of Σ in terms of the singularities of these maps was the *graphic* of the map, i.e. the image in the target of the singular locus. So far, we have only paid attention to the local behavior of this graphic, but the graphic itself makes sense globally. In fact, the global graphic will play key role in the next section. In this section we make the first steps toward understanding

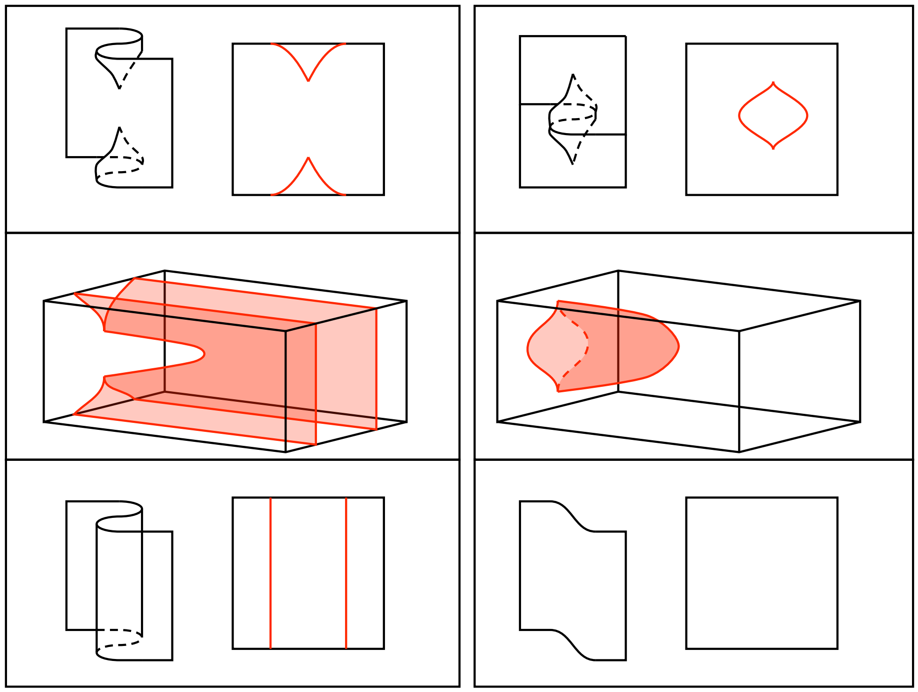
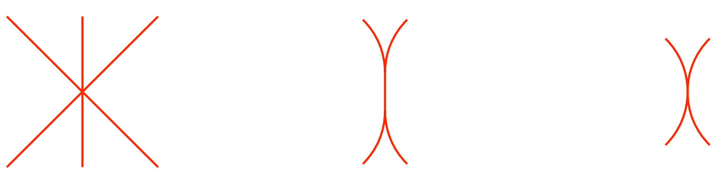


Figure 2.11: Cusp Inversions and Their Graphics

its structure.

By jet transversality, we know that each of the singular loci occurs on a submanifold of Σ or $\Sigma \times I$ and that these submanifolds map into the target via an immersion.[1] They are not typically embedded. In fact, we saw that in any neighborhood of the image of a Swallowtail singularity, there must be an arc where the images of two Fold loci intersect. In this section we want to better understand how these immersed manifolds intersect. For example, consider the following three scenarios:



(2.4.1)

These curves are not in general position, but from the analysis we have done so far, such configurations cannot be ruled out.

However, we can force our singular loci to be in general position by enlarging our

[1]This is true, at least if we restrict to the Local Diffeomorphism, Fold, Cusp, and 2D Morse singularities, in the 2-dimensional case, and restrict to the eight singularities listed in Table 2.3 in the 3-dimesnional case.

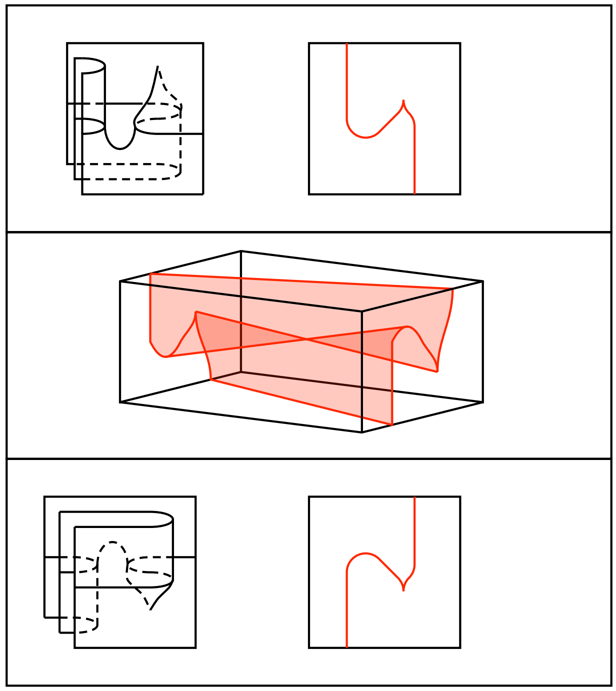


Figure 2.12: Cusp Flips and Their Graphics

collection of singularity strata and requiring (jet) transversality with respect to these strata. The new strata will live in the multi-jet spaces, and transversality to these strata will imply, among other things, that the images of the singular loci meet in general position. After introducing these strata, the Multi-jet Transversality Theorem 2.1.8 ensures that a generic set of maps satisfy both these new transversality conditions, as well as the previous ones.

Recall that the multi-jet space is defined as the pull-back of $J^{\mathbf{k}}(X,Y) = J^{k_1}(X,Y) \times \cdots \times J^{k_s}(X,Y)$ to $X^{(s)} = \{(x_1,\ldots,x_s) \in X^s \mid x_i \neq x_j \text{ for } 1 \leq i < j \leq s\}$, where $\mathbf{k} = (k_1,\ldots,k_s)$ is a multi-index of natural numbers. The multi-jet of a map $f : X \to Y$ is then given by,

$$j^{\mathbf{k}} f(x_1,\ldots,x_s) = (j^{k_1} f(x_1),\ldots,j^{k_s} f(x_s)).$$

Let $\Delta Y \subset Y^s$ denote the diagonal, and let $\beta : J^{\mathbf{k}}(X,Y) \to Y^s$ be the canonical projection. Then $\beta^{-1}(\Delta Y)$ is a sub-manifold of $J^{\mathbf{k}}(X,Y)$ which frequently use.

SpecialIzing to the case $X = \Sigma^2$ and $Y = \mathbb{R}^2$, consider the following submanifolds

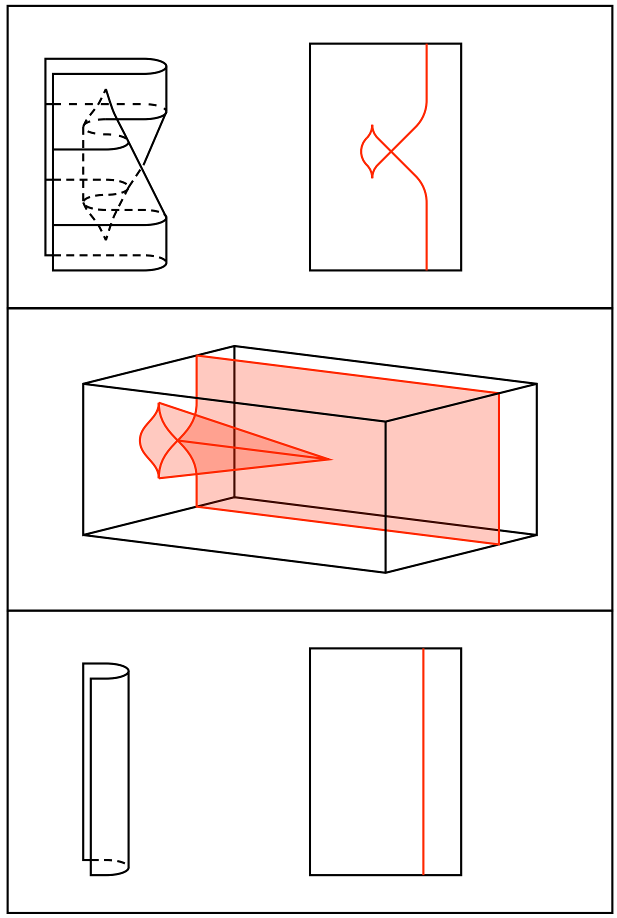


Figure 2.13: Swallowtails and Their Graphics

of the multi-jet spaces:

$$(S_{[11]} \times S_{[11]}) \cap \beta^{-1}(\Delta Y) \subset J^{(1,1)}(\Sigma, \mathbb{R}^2)$$
$$(S_{[11]} \times S_{[01]}) \cap \beta^{-1}(\Delta Y) \subset J^{(1,1)}(\Sigma, \mathbb{R}^2)$$
$$(S_{[01]} \times S_{[01],1}) \cap \beta^{-1}(\Delta Y) \subset J^{(1,2)}(\Sigma, \mathbb{R}^2)$$
$$(S_{[01],0} \times S_{[01],0}) \cap \beta^{-1}(\Delta Y) \subset J^{(2,2)}(\Sigma, \mathbb{R}^2)$$

These have codimensions 6, 5, 5, and 4, respectively and since the dimension of $\Sigma^{(2)}$ is $2+2=4$, only the last singularity occurs. Moreover we can consider the stratum,

$$(S_{[01]} \times S_{[01]} \times S_{[01]}) \cap \beta^{-1}(\Delta Y) \subset J^{(1,1,1)}(\Sigma, \mathbb{R}^2).$$

The codimension of this submanifold is 7, while the dimension of $\Sigma^{(3)}$ is 6, so this singularity

does not occur.

What does this mean? It means that for a generic map $f : \Sigma^2 \to \mathbb{R}^2$, whose multi-jets are transverse to all these strata, the only singularities that can have the same critical value (i.e. image in $\mathbb{R}^2$) are fold singularities, and that these must occur in pairs and not in triples. Moreover (if Σ is compact, which we will tacitly assume from now on) the critical values corresponding to multiple fold singularities must be isolated, and only a finite number occur. This rules out the first two scenarios from 2.4.1, but the third still remains.

The characteristic that distinguishes this final scenario from general position is that the tangent vectors to both curves coincide at the intersection point. This is in contrast to the intersection point which occurs, for example, in a neighborhood of a swallowtail critical value, in which the tangent vectors are transverse, see Figure 2.13. In order to rule out this scenario, we will have to understand how to read off these tangent vectors from the singularity strata.

Recall that the singularity stratum $S_{[01]}$ was defined as an open subset of the Thom-Boardmann stratum $S_1 \subset J^1(\Sigma, \mathbb{R}^2)$. It will be enough to work with the simpler Thom-Boardman stratum S_1. Recall that over S_1, there existed two canonical line bundles, $K = \ker df$ and $L = \operatorname{coker} df$ (where f is any local map representing the jet in S_1). Recall also, that $S_1^{(2)}$ denotes the inverse image of S_1 in the second jet space $J^2(\Sigma, \mathbb{R}^2)$, and that there is a map of bundles over S_1,

$$S_1^{(2)} \to \mathrm{Hom}(T\Sigma, TJ^1(\Sigma, \mathbb{R}^2)) \to \mathrm{Hom}(T\Sigma, TJ^1(\Sigma, \mathbb{R}^2)/TS_1)$$

where these vector bundles should be interpreted as their pullbacks to S_1.

Let $S_1(f)$ denote those points in Σ in which f obtains an S_1-singularity. The tangent space to $S_1(f) = (j^1 f)^{-1}(S_1)$ in Σ will be the kernel of the induced map,

$$T\Sigma \to TJ^1(\Sigma, \mathbb{R}^2)/TS_1 = \nu S_1 \tag{2.4.2}$$

which can be read off from the second jet, $j^2 f$. We would then like to use these vector spaces to obtain a stratification of $(S_1 \times S_1) \cap \beta^{-1}(\Delta Y)$, in particular by stratifying it according to whether the images of these vector spaces under df coincide or not. The problem is that the kernel to (2.4.2) does not form a vector bundle over $S_1^{(2)}$. The dimensions of these vector spaces jump.

Fortunately there is an open subset, $S_1^{(2),0} \subset S_1^{(2)}$, which has the minimal corank of zero (i.e. where the map (2.4.2) is surjective). Over this subspace, the kernel, C, forms a line bundle and we may try to proceed with our construction as planned. Note that the condition that $j^1 f \pitchfork S_1$ at $x \in \Sigma$ is precisely that the above map is surjective, i.e. that $j^2 f(x) \in S_1^{(2),0}$. This is why it is sufficient just to work with $S_1^{(2),0}$ and to ignore the rest of the stratum $S_1^{(2)}$.

Thus the space $(S_1^{(2),0} \times S_1^{(2),0}) \cap \beta^{-1}(\Delta Y)$ has a canonical pair of line bundles over it. We would like to map these line bundles to TY and form a further stratification according whether their images coincide. We run into a similar problem in that the images in TY do not form a vector bundle over $(S_1^{(2),0} \times S_1^{(2),0}) \cap \beta^{-1}(\Delta Y)$, again because their dimensions may jump.

Determining the dimension of the image of C under df is equivalent to determining the dimension of the intersection of C with $K = \ker df$. Fortunately we have already stratified $S_1^{(2)}$ according to this dimension. This is precisely the corank of the induced map,

$$K \to TJ^1(\Sigma, \mathbb{R}^2)/TS_1 = \nu S_1 \cong \mathrm{Hom}(K, L).$$

The only relevant multi-jet stratum will be,

$$S_{1,0\times 1,0} := [(S_1^{(2),0} \cap S_{1,0}) \times (S_1^{(2),0} \cap S_{1,0}))] \cap \beta^{-1}(\Delta Y)$$

over which there exists a pair of lines C_1 and C_2, whose images under df form another pair of lines $df(C_1), df(C_2) \subset TY$. The map from $S_{1,0\times 1,0}$ to the Grassmannian of pairs of lines is a submersion, which can be seen after choosing local coordinates. This Grassmannian is stratified into a codimension zero stratum and a codimension one stratum according to whether these lines coincide or not. Hence we get an induced stratification of $S_{1,0\times 1,0}$ by pulling-back the stratification of this Grassmannian, and hence a pair of submanifolds of $J^{(2,2)}(\Sigma, \mathbb{R}^2)$ of total codimensions 4 and 5 respectively. The later type of singularity does not occur as $\Sigma^{(2)}$ only has dimension 4. This rules out the last scenario of 2.4.1 for generic maps $f : \Sigma \to \mathbb{R}^2$. We summarize these results in the following theorem.

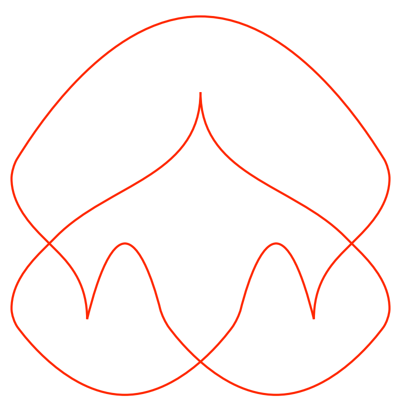
Theorem 2.4.3. *The graphic of a generic map $f : \Sigma^2 \to \mathbb{R}^2$, with Σ^2 compact, consists of a finite number of embedded curves (the $S_{[01],0}$-singularities) whose projections to $\mathbb{R}$ are diffeomorphisms, and two finite collections of isolated points (corresponding to $S_{[11],0}$- and $S_{[01],1,0}$-singularities), such that in a neighborhood of each of these points there exist coordinates in which the graphic has the form of a 2D Morse graphic (Figure 2.6) or the form*

of a Cusp graphic (Figure 2.7), respectively. Moreover, the embedded curves may intersect, but are in general position.

It will be useful if we codify these sorts of diagrams in an abstract definition. We will choose to call this new sort of diagram a *graphic*, which generalizes our previous use of terminology. Then the above theorem may be expressed by saying that the images of the critical points of a generic function form an abstract graphic.

Definition 2.4.4. A 2-dimensional *graphic* is a diagram in $\mathbb{R}^2$ consisting of a finite number of embedded curves whose projections to $\mathbb{R}$ are diffeomorphisms, and two finite collections of isolated points. These curves and points are labeled. The curves are labeled by $S_{[01],0}$ together with one of the two possible indices of the fold singularities. The points are labeled by either $S_{[11],0}$ or $S_{[01],1,0}$, together with the possible indices of these singularities. We require that in a neighborhood of each of the points there exist coordinates in which the graphic has the form of a 2D Morse graphic (Figure 2.6) or the form of a Cusp graphic (Figure 2.7), respectively. Moreover, the embedded curves are required to be in general position. ◇

Example 2.4.5. The following is one possible graphic for a generic map from $\Sigma = \mathbb{RP}^2$ to $\mathbb{R}^2$.



◇

The intersections of the 3-dimensional singular loci can similarly be controlled by providing an appropriate multi-jet stratification. In this case the stratification and resulting phenomena are more interesting. We will again begin with a course analysis that will rule out many possible situations. Then we will provide a finer stratification of the remaining

cases, taking into account tangential information, as we did in the 2-dimensional case. Recall the seven non-tirival singularities listed in Table 2.3, with their codimensions.

The codimension of $S_a \times S_b \cap \beta^{-1}(\mathbb{R}^2 \times I)$ is given by,

$$\operatorname{codim} S_a + \operatorname{codim} S_b + 3$$

while the dimension of $(\Sigma \times I)^{(2)}$ is 6. By comparing the codimensions of the relevant strata we see that there are only the following three possibilities with codimension ≤ 6,

$$\begin{aligned} &S_{[001],0} \times S_{[001],[01],0} \cap \beta^{-1}(\mathbb{R}^2 \times I) && \text{codim} = 6 \\ &S_{[001],0} \times S_{[011],0} \cap \beta^{-1}(\mathbb{R}^2 \times I) && \text{codim} = 6 \\ &S_{[001],0} \times S_{[001],0} \cap \beta^{-1}(\mathbb{R}^2 \times I) && \text{codim} = 5. \end{aligned}$$

Moreover, the dimension of $(\Sigma \times I)^{(3)}$ is 9, and hence $S_{[001],0} \times S_{[001],0} \times S_{[001],0} \cap \beta^{-1}(\mathbb{R}^2 \times I)$, of codimension 9 in $J^{(2,2,2)}(\Sigma \times I, \mathbb{R}^2 \times I)$, is the only triple intersection which can occur.

We must now analyze the tangential structure of all these singularity loci. As before, for each of the singularities S_α, where $\alpha =$ [001 or [011], we may look at it's pre-image $S_\alpha^{(2)}$ in $J^2(\Sigma \times I, \mathbb{R}^2 \times I)$, which forms a bundle over S_α. Again there are maps of fiber bundles over S_α,

$$S_\alpha^{(2)} \to \operatorname{Hom}(T(\Sigma \times I), TJ^1(\Sigma \times I, \mathbb{R}^2 \times I)) \to \operatorname{Hom}(T(\Sigma \times I, \nu S_\alpha),$$

where each vector bundle is considered a bundle over S_α via pulling it back.

We can consider the open subset of those maps $TX \to \nu S_\alpha$ consisting of minimal corank (i.e. maximal rank). The inverse image defines an open subset $S_\alpha^{(2),0} \subseteq S_\alpha^{(2)}$. Over $S_\alpha^{(2),0}$ there is a vector bundle C_α consisting of the kernel of the map, $TX \to \nu S_\alpha$. If $f : \Sigma \times I \to \mathbb{R}^2 \times I$ is generic and takes an S_α-singularity at $x \in X$, then $j^1 f \pitchfork S_\alpha$ at x, which is equivalent to $j^1 f(x) \in S_\alpha^{(2),0} \subseteq S_\alpha^{(2)}$. If $S_\alpha(f)$ denotes the submanifold of $\Sigma \times I$ consisting S_α-singularities, then the tangent space to $S_\alpha(f)$ at x is precise the vector space $C_{\alpha, j^2 f(x)} \subset T_x(\Sigma \times I)$.

We will want to consider the image of C_α under the map $df : TX \to TY$. Again this is not a bundle on all of $S_\alpha^{(2),0}$, but we have already stratified $S_\alpha^{(2)}$ into substrata on which df has a fixed dimensional intersection with C_α, and on the intersections of this stratification with the open set $S_\alpha^{(2),0}$, we get vector bundles $df(C_\alpha)$. The possible cases are listed in Table 2.6.

Table 2.6: Tangent Spaces to the Images of 3D Singularities

Singularity (α)	Dim. of C_α	Dim. of $df(C_\alpha)$	Properties
$S_{[001],0}$	2	2	Proj. to $T(\mathbb{R} \times I)$ is surjective.
$S_{[001],[01],0}$	1	1	
$S_{[011],0}$	1	1	Does not contain ∂_{y_0}

Now we may use these bundles on the multi-jet strata, together with the subbundles $\mathrm{Span}\{\partial_{y_0}\}$ and $\mathrm{Span}\{\partial_{y_0}, \partial_{y_1}\}$, to construct a further refinement of this stratification. Transversality with respect to this stratification will then imply that the images of the singular loci are in general position with respect to one another and with respect to the subbundles $\mathrm{Span}\{\partial_{y_0}\}$ and $\mathrm{Span}\{\partial_{y_0}, \partial_{y_1}\}$.

On the multi-jet stratum $S_{[001],0} \times S_{[001],0}$ (intersected with $S^{(2),0}_{[001]} \times S^{(2),0}_{[001]}$), we have the two vector bundles denoted by $df(C_{[001]})$. We will further stratify $S_{[001],0} \times S_{[001],0}$ by the dimension of the intersection of these 2-dimensional bundles, as well as their intersection with the plane, $\ker dp^2 : T(\mathbb{R}^2 \times I) \to T(I)$. There is a codimension 2 substratum, where these vector bundles coincide in $T(\mathbb{R}^2 \times I)$. The codimension zero substratum consists of where they only have a 1-dimensional intersection. Inside this stratum, there is an additional sub-stratification according to whether this intersection is contained in $\mathrm{Span}\{\partial_{y_0}, \partial_{y_1}\}$, or not. See Table 2.7. Note that $\dim df(C_{\alpha_1}) \cap df(C_{\alpha_2}) \cap \mathrm{Span}\{\partial_{y_0}\} = (0)$ automatically.

Next consider the multi-jet stratum $S_{[001],0} \times S_{[011],0}$. Over this stratum, (intersected with $S^{(2),0}_{[001]} \times S^{(2),0}_{[011]}$), we have a line bundle and a rank two vector bundle $C_{[011]}$ and $C_{[001]}$, respectively. We will stratify this according to the whether their images under df intersect each other. There is a an open stratum in which they do not intersect and a codimension one stratum in which they do. See Table 2.7.

Now consider the multi-jet stratum $S_{[001],0} \times S_{[001],[01],0}$ intersected with the stratum $S^{(2),0}_{[001]} \times S^{(2),0}_{[011]}$. Over this intersection we have two rank two vector bundles C_1 and C_2, whose images $df(C_1)$ and $df(C_2)$ form a rank two vector bundle and a line bundle, respectively. We will further stratify according to the dimension of the intersection of $df(C_1)$ and $df(C_2)$. There is an open substratum where they intersect trivially, and a codimension one stratum where they intersect non-trivially. See Table 2.7.

Finally over the intersection of $S^{(2),0}_{[001]} \times S^{(2),0}_{[001]} \times S^{(2),0}_{[001]}$ with $S_{[001],0} \times S_{[001],0} \times S_{[001],0}$ there are three rank two vector bundles $df(C_1)$, $df(C_2)$, and $df(C_3)$. We will further stratify

Table 2.7: Codimensions of 3D Multi-jet Strata

Singularity	A	B	C	Total Codimension
$S_{[001],0} \times S_{[001],0} \cap \beta^{-1}(\mathbb{R}^2 \times I)$	2	0		5+2 = 7
$S_{[001],0} \times S_{[001],0} \cap \beta^{-1}(\mathbb{R}^2 \times I)$	1	0	1	5 + 1 = 6
$S_{[001],0} \times S_{[001],0} \cap \beta^{-1}(\mathbb{R}^2 \times I)$	1	0	0	5
$S_{[001],0} \times S_{[011],0} \cap \beta^{-1}(\mathbb{R}^2 \times I)$	0	0	0	6
$S_{[001],0} \times S_{[011],0} \cap \beta^{-1}(\mathbb{R}^2 \times I)$	1			6 + 1 = 7
$S_{[001],0} \times S_{[001],[01],0} \cap \beta^{-1}(\mathbb{R}^2 \times I)$	0	0	0	6
$S_{[001],0} \times S_{[001],[01],0} \cap \beta^{-1}(\mathbb{R}^2 \times I)$	1			6 + 1 = 7

$$A = \dim df(C_{\alpha_1}) \cap df(C_{\alpha_2}),$$
$$B = \dim df(C_{\alpha_1}) \cap df(C_{\alpha_2}) \cap \mathrm{Span}\{\partial_{y_0}\}, \text{ and}$$
$$C = \dim df(C_{\alpha_1}) \cap df(C_{\alpha_2}) \cap \mathrm{Span}\{\partial_{y_0}, \partial_{y_1}\}.$$

according to the dimension of their intersection in TY. There is a open substratum in which none of these 2-planes intersect. There is a codimension two substratum where two of the 2-planes coincide and there is a codimension 4 substratum in which all the 2-planes coincide. The total codimension of these strata are 9, 11, and 13, respectively, and since the dimension of $(\Sigma \times I)^{(3)}$ is 9, only the first occurs.

Motivated by the above analysis of 3-dimensional multi-jet singularities, we introduce the following abstract definition of a 3-dimensional graphic. A generic map $f : \Sigma \times I \to \mathbb{R}^2 \times I$ then induces such a graphic.

Definition 2.4.6. A 3-dimensional *graphic* is a diagram in $\mathbb{R}^2 \times I$ (with standard coordinates (y_0, y_1, y_2)) consisting of a finite collection of labeled embedded points, arcs, and surfaces, such that the following conditions are met:

- The points are labeled by one of the following four labels: $S_{[011],[100],0}$, $S_{[001],[11],0}$, $S_{[011],[001],0}$, and $S_{[001],[01],1,0}$, together with an appropriate index.
- The arcs are labeled by either $S_{[011],0}$ or $S_{[001],[01],0}$, an appropriate index, and for each arc the projection to I is a local diffeomorphism.
- The surfaces are labeled by $S_{[001],0}$, an appropriate index, and for each surface the projection to $\mathbb{R} \times I$ is a local diffeomorphism.

– In a neighborhood of each point in each labeled arc, there exist coordinates in which the arrangement of arcs and surfaces matches the arrangement of arcs and surfaces induced by the images of singular loci associated to $S_{[011],0}$ and $S_{[001],[01],0}$ singularities. See Propositions 2.2.22 and 2.2.21 and Figure 2.9.

– In a neighborhood of each labeled point there exist coordinated in which the arrangement of arcs and surfaces matches the images of the singular loci associated to $S_{[011],[100],0}$, $S_{[001],[11],0}$, $S_{[011],[001],0}$, and $S_{[001],[01],1,0}$ singularities. See Propositions 2.2.19 2.2.17, 2.2.18, and 2.2.15 and Figures 2.10, 2.11, 2.12, and 2.13.

– The arcs, points, and surfaces are in general position.

Thus there is a 1-dimensional submanifold in which precisely two surfaces intersect. We further require the the tangent space to this 1-dimensional intersection locus only intersect $\mathrm{Span}\{\partial_{y_0}, \partial_{y_1}\}$ non-trivially at a finite number of isolated points. We call these points *inversion points.* $\diamond$

Theorem 2.4.7. *Let Σ be a compact surface, and $f : \Sigma \times I \to \mathbb{R}^2 \times I$ a generic map. Then the images of the singular loci of f form a* graphic*, as in the above definition.*

2.5 The Planar Decomposition Theorem

In this section we prove the main theorems of this chapter. Building on the results already obtained, we introduce the notion of a planar diagram. A planar diagram is essentially an arrangement of arcs and points in $\mathbb{R}^2$, together with a certain amount of combinatorial data. A generic map from a surface Σ to $\mathbb{R}^2$ leads to a planar diagram, and moreover the surface can be completely recovered from the diagram, up to diffeomorphism. Similarly, we introduce the notion of a spacial diagram, which plays the same role for maps from $\Sigma \times I$ to $\mathbb{R}^2 \times I$. Two planar diagrams are defined to be equivalent if there is a spacial diagram extending the planar diagrams. We show, among other results, that equivalence classes of planar diagrams are in natural bijection with diffeomorphism classes of surfaces.

By the Thom transversality theorem, we know that there exist generic maps from any surface, Σ, into $\mathbb{R}^2$. What we have been studying so far has essentially been the local structure of these maps, and this gives us a complete, but local, understanding of our surface Σ. Some progress was made towards a global understanding in the last section, where we

were able to completely understand the global structure of the graphic of a generic map. In this section we see how we can use the graphic assemble these local descriptions into a global description of Σ.

As we have done before, we will begin by describing the analogous situation one dimension lower. Suppose that Y is a 1-dimensional manifold, equipped with a Morse function, f, as in Figure 2.14. For every point $r \in \mathbb{R}$ and every point $y \in Y$, such that $f(y) = r$, we know that there exist neighborhoods around y and r in which f takes a particularly nice form. In particular f will be a local diffeomorphism if r is not a critical value for f, and if r is a critical value for f, then there is precisely one critical point $y \in Y$ whose critical value is r. Around this point there are standard Morse coordinates in which f takes the form of Equation 2.2.1. Around any other point in $f^{-1}(r)$, f is a local diffeomorphism.

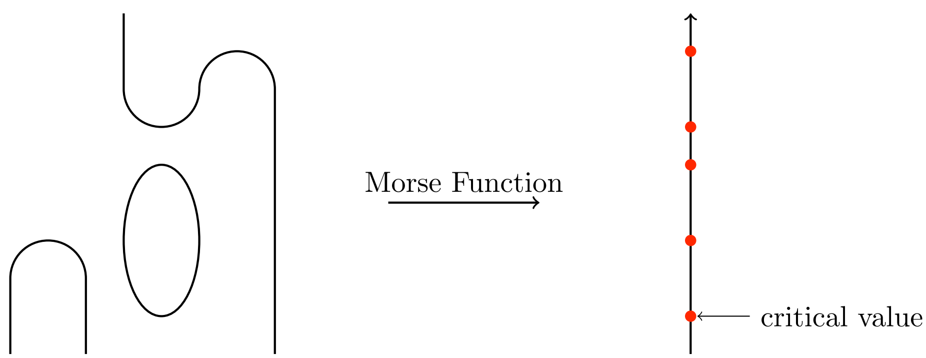
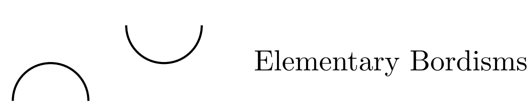


Figure 2.14: A Morse Function for a 1-Manfold

Thus we may choose open neighborhoods covering $\mathbb{R}$ in such a way that the inverse images of these open sets decompose Y into pieces, U_i, which consist of either several sheets mapping by a local diffeomorphism onto Y, or the disjoint union of such sheets with a single *elementary bordism.*



In this manner we may write Y as the union $\cup U_i$, compare Figures 2.14 and 2.15.

We may choose these open sets in $\mathbb{R}$ to be connected and to be as small and as well-shaped as we like. Those pieces which contain no critical points, and hence are simply a collection of sheets, may be (canonically) identified with $S_i \times f(U_i)$, where S_i is the set

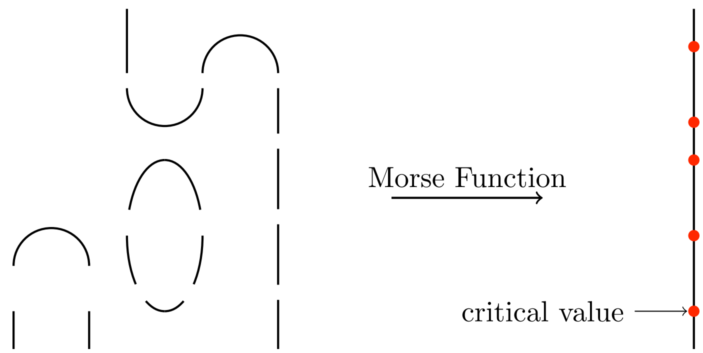


Figure 2.15: A Linear Decomposition of a 1-Manifold

of sheets and $f(U_i)$ is the image of U_i in $\mathbb{R}$. The pieces which contain an isolated critical point have a similar description.

Given the decomposition of $\mathbb{R}$, together with sheets S_i for each region, we would like to be able to reconstruct Y. For this we need to know how to glue the resulting U_i together. The overlaps of any two U_i are regions which contain no critical values, and hence give rise to a transition function $S_i \to S_j$, which is an isomorphism of sets. Since $\mathbb{R}^1$ has covering dimension one, we may assume that all triple intersections are empty.

For convenience, we can trivialize these sets S_i as follows. Let $\mathbf{N}$ denote the ordered set $(1\ 2\ \ldots\ N)$. We trivialize the sets S_i by choosing an isomorphism $S_i \cong \mathbf{N}$, for some N. The elementary bordisms corresponding to critical points involve precisely two sheets, and so are determined by a set S_i with a distinguished pair of elements and a labeling of "$\subset$" or "$\supset$" depending on the two possible indices of the critical point. We can then choose an identification $S_i \cong \mathbf{N}$ in which the distinguished pair of elements is $\{N, N-1\}$. Thus the two boundaries of such a region have sheets which are identified with either $\mathbf{N}$ or $\mathbf{N-2}$.

Together with these choices, a Morse function $f : Y \to \mathbb{R}$ gives us a combinatorial description of Y. We get a diagram on $\mathbb{R}$ by dividing it into regions labeled by either $(\mathbf{N})$, $(\mathbf{N},\subset)$, or $(\mathbf{N},\supset)$. The boundaries of these regions are labeled by permutations of the appropriate sets. Figure 2.16 gives an example of such a *linear diagram* based off of the Morse function depicted in Figure 2.14.

Conversely, given a linear diagram one can reconstruct a 1-manifold together with a Morse function inducing the original diagram. This 1-manifold can be constructed by gluing the basic pieces together exactly as the linear diagram dictates. We invite the reader

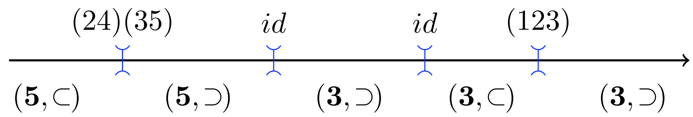


Figure 2.16: A Linear Diagram giving a Combinatorial Description of a 1-Manifold

to check that Figure 2.16 gives a prescription which exactly reproduces the 1-manifold of Figure 2.14, up to diffeomorphism.

To complete our description of 1-manifolds in terms of linear diagrams, we must consider when two diagrams correspond to the same 1-manifold. This is where Cerf theory provides an essential tool. Any two morse functions on the same manifold Y may be connected by a path of functions, which we consider as a map, $Y \times I \to \mathbb{R} \times I$. We have seen how to derive normal coordinates for these sorts of functions, as in Propositions 2.2.9, 2.2.10, and 2.2.11, and in this way we obtain a completely local description of such functions.

Just as a Morse function, together with suitable choices, provided a diagram in the target $\mathbb{R}$, so too a generic map $Y \times I \to \mathbb{R} \times I$ provides a diagram in the target $\mathbb{R} \times I$, which we termed the graphic of the map. We build a diagram based on the graphic. Our local description ensures that we may choose a covering of $\mathbb{R} \times I$ by open sets so that in each open set f has a particular normal form, and again the inverse images over each such region are determined up to canonical isomorphism by a certain amount of combinatorial data.

The covering dimension of $\mathbb{R} \times I$ is two, and so we may choose the covering so there are at most triple intersections. Moreover, we may arrange for our cover to be as small and nicely-shaped as we desire. If we start with linear diagrams for the initial and final Morse functions, we may choose the cover to be compatible with the covers of $\mathbb{R}$ at each end of $\mathbb{R} \times I$. For a linear diagram, the transitions between open sets were labeled by the induced isomrphism of combinatorial data, and in this 2-dimensional diagram it is no different. On any double intersection of the cover we have a label by an isomorphism of the sheet data. On triple intersections a cocycle condition must be satisfied. We will make this more precise in the analogous case relevant for surfaces.

In the end, one obtains a notion of *planar diagram*, analogous to the linear diagrams considered previously. Out of such a diagram, one obtains a manifold $Y \times I$ and a map

to $\mathbb{R} \times I$, inducing the original diagram, and each diagram is built out of a finite number of simple pieces. These provide the relations necessary to pass from one linear diagram to another. Call two linear diagrams *equivalent* if there is a planar diagram extending both. Leaving aside the details, one obtains a pair of theorems as below.

Theorem 2.5.1. *Two linear diagrams are equivalent if and only if they can be related by a finite sequence of the following moves:*

1. *Isotopy.*
2. *If* $\sigma \in S_{N-2} \subset S_N$*, then* $\sigma(\mathbf{N},\subset) = (\mathbf{N},\subset)\sigma$ *and* $\sigma(\mathbf{N},\supset) = (\mathbf{N},\supset)\sigma$*.*
3. $(\mathbf{N},\subset) = (\mathbf{N},\subset)(N\ N-1)$ *and* $(N\ N-1)(\mathbf{N},\supset) = (\mathbf{N},\supset)\sigma$*.*
4. $\sigma_1(\mathbf{N})\sigma_2 = \sigma_1 \cdot \sigma_2$*.*
5. $(\mathbf{N},\subset)(N-2\ N-1\ N)(\mathbf{N},\supset) = id$*.*

Theorem 2.5.2. *Here "suitable choices" refers to the choice of covering of* $\mathbb{R}$ *and associated combinatorial data.*

1. *A Morse function* $f : Y^1 \to \mathbb{R}$*, together with suitable choices, produces a linear diagram.*
2. *Any 1-manifold admits a Morse function and suitable choices.*
3. *To each linear diagram there is a 1-manifold, together with a Morse function and suitable choices, inducing the given linear diagram.*
4. *Given a 1-manifold,* Y*, two Morse functions on* Y *with suitable choices induce equivalent linear diagrams.*
5. *Given a 1-manifold,* Y*, there exists a Morse function and suitable choices for the Morse function, and consequently there exists a corresponding linear diagram. To this diagram, then corresponds a 1-manifold,* Y' *(also equipped with a Morse function reproducing the linear diagram). Then* Y' *is isomorphic to* Y*.*
6. *Two linear diagrams induce isomorphic manifolds if and only if they are equivalent.*

In particular the isomorphism classes of 1-manifolds are in natural bijection with the equivalence classes of linear diagrams.

We will not provide proofs, since these statements can be proved in a manner similar to what follows. The intrepid reader might try to prove them on her own. These two theorems serve as prototypes for higher dimensional versions of these results, to which we now turn.

In dimension two we are concerned not with Morse functions, but with maps to the plane, $\mathbb{R}^2$. We have already analyzed the singularities of these maps, and using this analysis we will, with some choices involved, be able to produce a planar diagram. This diagram will be labeled with certain combinatorial data, and is itself a combinatorial object, but nevertheless we will be able to reconstruct the manifold from the diagram. Two maps to $\mathbb{R}^2$ can be connected by a path of maps, which we consider as a single map $\Sigma \times I \to \mathbb{R}^2 \times I$, and which we can replace by a generic map. Again we have analyzed the the singularities of such maps. This allows us to construct 3-dimensional diagrams which we use to define when two planar diagrams are equivalent. The surfaces corresponding to two planar diagrams will be isomorphic if and only if the planar diagrams are equivalent.

Fix a surface Σ^2 and a generic map $f : \Sigma \to \mathbb{R}^2$. If $p \in \mathbb{R}^2$ is a point which is not a critical value of f, then f is a local diffeomorphism around p. Propositions 2.2.9, 2.2.10, and 2.2.11 allow us to choose normal coordinates around any point which is a critical value. Theorem 2.4.3 ensures that the images of the singular loci will form a graphic in $\mathbb{R}^2$. We will choose the normal coordinate neighborhoods in a way that is compatible with the graphic.

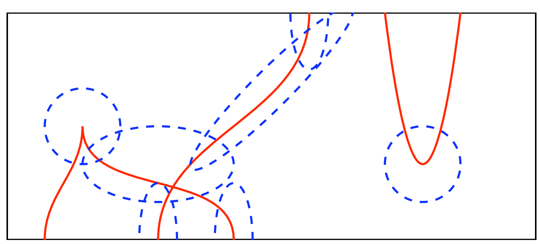


Figure 2.17: Choice of Local Neighborhoods Decomposes $\mathbb{R}^2$

The covering dimension of $\mathbb{R}^2$ is two, and hence we may assume that these normal coordinate neighborhoods have at most triple intersections. Moreover, we may choose them so that in these triple intersections, there are are no critical values. Similarly, we may assume that the double intersections either contain no other critical points, or contain a portion of a single arc in the graphic, see Figure 2.18.

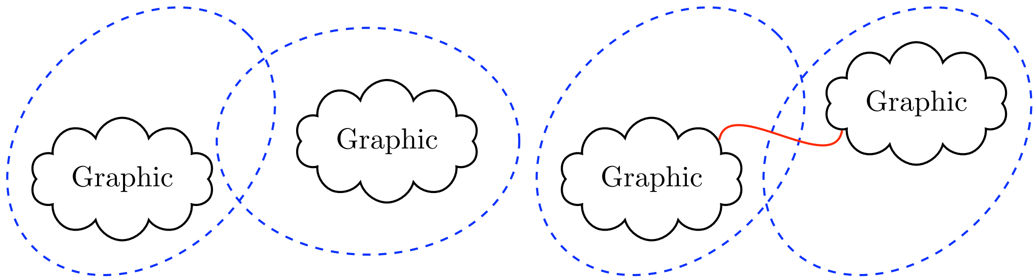
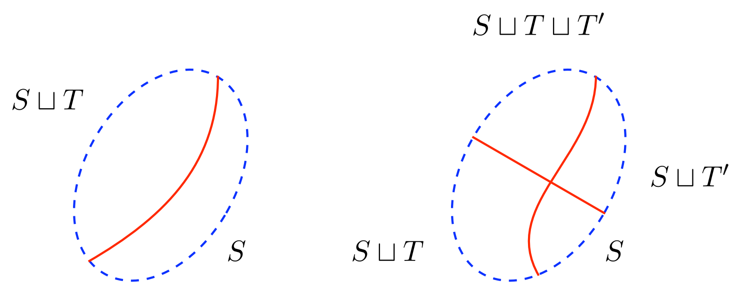


Figure 2.18: Possible Normal Coordinate Intersections

Thus each normal coordinate neighborhood contains either an isolated 2D Morse singularity, an isolated Cusp singularity, an isolated point where precisely two fold singularities occur (and in which the fold loci intersect in general position), a single arc of fold singularities, or no critical values at all. If U is a region which contain no critical vales at all, then $f : f^{-1}(U) \to \mathbb{R}^2$ is a local diffeomorphism and hence $f^{-1}(U)$ consists of a disjoint union of "sheets". As before, we have a canonical isomorphism $f^{-1}(U) \cong S \times U$, where S is the set of sheets, i.e. the components of $f^{-1}(U)$.

If U is a region containing a portion of a single arc of the graphic, then that arc divides U into two components, U_0 and U_1. This in turn divides $f^{-1}(U)$ into two components, and away from the singular fold arc, f restricts on $f^{-1}(U)$ to a local diffeomorphism. Thus, on each component we have a canonical identification via f with $S_i \times U_i$, where S_i is the set of sheets over U_i. S_0 and S_1 differ by exactly two elements, in the sense that there is a two element set T and a canonical identification $S_0 \cong S_1 \sqcup T$ or a canonical identification $S_0 \sqcup T \cong S_1$.

If U is a region which contains a point where two fold arcs cross, then the fold arcs divide U into four regions. A similar analysis of sheet data applies to this situation. In this case there exist two 2-element sets T and T', and four regions have sheet data which can be identified with the sets S, $S \sqcup T$, $S \sqcup T'$, and $S \sqcup T \sqcup T'$, respectively. The sheet data associated to a 2D Morse singularity is essentially the same as the data associated to the fold singularity.



If U is a region containing a cusp singularity, then there exist two arcs of fold singularities within U. These divide U into two regions, which have their associated sheet data S and S'. Each of the fold singularities also has sheet data $(S_0, S_0 \sqcup T_0)$ and $(S_1, S_1 \sqcup T_1)$, where T_0 and T_1 are two element sets. Moreover we have isomorphisms $S \cong S_0$, $S \cong S_1$, and $S_0 \sqcup T_0 \cong S' \cong S_1 \sqcup T_1$, and under these identifications, T_0 and T_1 share one element in common, $\{p\}$. Let $T_2 = T_0 \cup T_1 \setminus \{p\}$. Then the following diagram commutes,

$$\begin{tikzcd}
S \arrow[r] \arrow[d, "id"'] & S_0 \arrow[r, hook] & S_0 \sqcup T_0 \arrow[r] & (S' \setminus T_2) \sqcup T_2 \arrow[d, "id \sqcup \sigma"] \\
S \arrow[r] & S_1 \arrow[r, hook] & S_1 \sqcup T_1 \arrow[r] & (S' \setminus T_2) \sqcup T_2
\end{tikzcd}$$

where $\sigma : T_2 \to T_2$ is the non-trivial automorphism of this 2-element set. We call such data *cusp sheet data*, see Figure 2.19. It is determined by the sets S, S' and the two maps $S \rightrightarrows S'$.

The choice of normal coordinate neighborhoods allows us to break our surface Σ into a union of elementary pieces. The graphic together with the sheet data allows us to identify these elementary pieces with standard elementary pieces. In order to recover the surface Σ, however, we need to know have to glue these elementary pieces together. This gluing information is additional data we will assign to the regions where the normal coordinate neighborhoods intersect. As we have already remarked, the double intersections of the normal coordinate neighborhoods occur in two varieties. Either the double intersection contains no critical points, or it contains a portion of a single arc of fold critical points, see Figure 2.18. In the first case, where there are no critical points in the intersection, the gluing data amounts to an identification of the sheet data, and is thus given by an isomorphism of sets $S_0 \cong S_1$.

In the second case we need to understand how to glue together elementary pieces which correspond to single fold singularities. Let the normal coordinate neighborhoods be

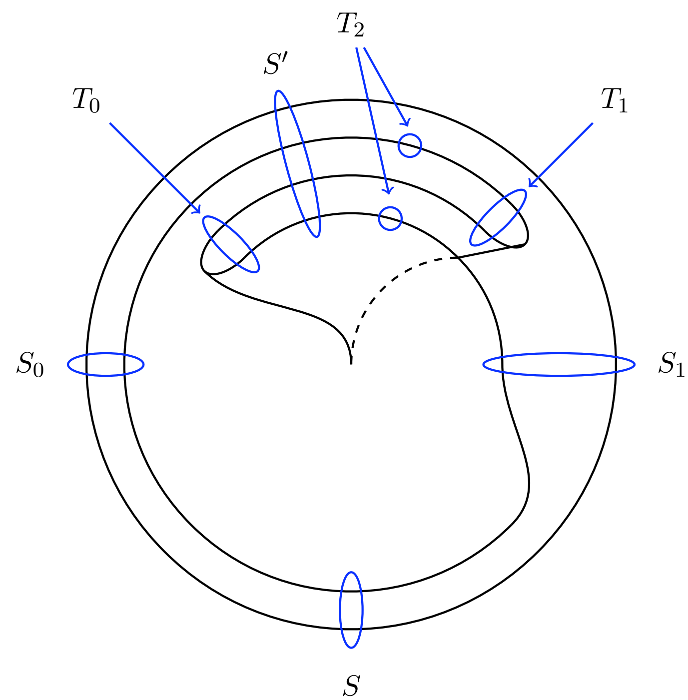


Figure 2.19: Cusp Sheet Data

U_0 and U_1. Recall that the sheet data for such singularities consists of a set of sheets S_i, and an additional pair of sheets T_i. Thus the fold locus splits, say, U_0 into two halves and the sheets over each half are identified with S_1 and $S_1 \sqcup T_1$, respectively. A straightforward calculation shows that a diffeomorphism $f^{-1}(U_0) \cong f^{-1}(U_1)$, respecting the map to $\mathbb{R}^2$ is equivalent to an isomorphism of pairs of sets $(S_0, T_0) \cong (S_1, T_1)$. We call such data fold gluing data.

Triple intersections of the normal coordinate neighborhoods only occur away from any critical points. On such triple intersections, the gluing data corresponding to the three adjacent double intersections must satisfy the obvious cocycle identity. With these considerations we introduce the following definition.

Definition 2.5.3. A *planar diagram* consists of a graphic in $\mathbb{R}^2$ (as in Definition 2.4.4), together with a covering by open sets, such that

1. Each open set in the covering is diffeomorphic to an open ball.

2. Each set in the covering either

Table 2.8: Planar Sheet Data and Gluing Data

Normal Coordinate Neighborhood	Sheet Data
No Crit. Points	a set S
Fold	(S, T), $\lvert T \rvert = 2$
2D Morse	(S, T), $\lvert T \rvert = 2$
Fold Crossing	(S, T_0, T_1), $\lvert T_i \rvert = 2$
Cusp	$(S \rightrightarrows S')$, cusp sheet data
Gluing Region	**Gluing Data**
No Crit. Points	$S_0 \cong S_1$
Fold	$(S_0, T_0) \cong (S_1, T_1)$

(a) contains no critical points,

(b) contains a single portion of a fold arc, and no other critical points,

(c) contains contains two portions of fold arcs and a single fold crossing point,

(d) contains a single 2D Morse critical point and only the two fold arcs associated with this singularity, or

(e) contains a single Cusp critical point and only the two fold arcs associated with this singularity.

In addition, each open set in the covering is equipped with appropriate sheet data as in Table 2.8. Thus in an given covering set U, and for any point $x \in U$ which is not a critical point, there is an associated set $S_{x,U}$, which is built out of the sheet data of U and the data of the graphic.

Furthermore, we require the double intersections of this covering to be connected and one of two kinds. Either the double intersection contains no critical points, or it contains a single portion of a fold arc. In these cases we assign appropriate gluing data to each double intersection, as per Table 2.8. Finally, we require that there are no 4-fold intersections and that triple intersections contain no critical points. The gluing data around a triple intersection is required to satisfy the obvious cocycle condition. ◇

Given a generic map of a surface Σ to $\mathbb{R}^2$, the results of the previous sections imply that we may choose coordinate neighborhoods covering $\mathbb{R}^2$, and after doing so we obtain

compatible sheet data and gluing data which, together with the graphic of the map, yield a planar diagram. Given a planar diagram, we may use the sheet data, gluing data, and the graphic to glue together standard elementary pieces, thereby constructing a surface. This surface comes naturally equipped with a smooth map to $\mathbb{R}^2$, which induces the original graphic underlying the planar diagram. To understand when two planar diagrams yield diffeomorphic manifolds, we must consider the analogous 3-dimensional situation.

Table 2.9: Spacial Sheet Data

Normal Coordinate Neighborhood	Sheet Data
No Crit. Points	S, set
Fold	(S, T), sets, $\lvert T\rvert = 2$
Fold Crossing	(S, T_0, T_1), sets, $\lvert T_i\rvert = 2$
Fold Crossing Inversion	(S, T_0, T_1), sets, $\lvert T_i\rvert = 2$
Fold Triple Crossing	(S, T_0, T_1, T_2), sets, $\lvert T_i\rvert = 2$
2D Morse	(S, T), sets, $\lvert T\rvert = 2$
2D Morse $\cap$ Fold	(S, T_0, T_1), sets, $\lvert T_i\rvert = 2$
Cusp	$(S \rightrightarrows S')$, Cusp Data
Cusp $\cap$ Fold	$(S \rightrightarrows S', T)$, Cusp Data and set, $\lvert T\rvert = 2$
2D Morse Relation	(S, T), sets, $\lvert T\rvert = 2$
Cusp Inversion	$(S \rightrightarrows S')$, Cusp Data
Cusp Flip	$(S \rightrightarrows S')$, Cusp Data
Swallowtail	$(S \Rrightarrow S')$, Swallowtail Data

Any two generic maps from a surface Σ to $\mathbb{R}^2$ may be connected by a generic map $\Sigma \times I \to \mathbb{R}^2 \times I$. The results of the previous sections imply that we can choose coordinate neighborhoods in a manner which is compatible with the graphic of this map. We must determine the additional sheet data need to Identify the inverse images of these coordinate neighborhoods with standard elementary pieces. By the results of the last section, we may assume that each normal coordinate neighborhood is one of thirteen possible types, as listed in Table 2.9. The analysis of the sheet data then occurs just as it did in the 2-dimensional setting.

The no critical point, Fold, 2D Morse, Fold Crossing, Fold Crossing Inversion,

Cusp, and Cusp Inversion normal coordinate neighborhoods have sheet data which is exactly identical to their 2-dimensional analogs. The Fold triple crossing sheet data is analogous to the fold crossing sheet data. The three sheets of folds divide the normal coordinate neighborhood into eight regions, and these regions have sheet assignments S, $S \sqcup T_i$, $S \sqcup T_i \sqcup T_j$, or $S \sqcup T_0 \sqcup T_1 \sqcup T_2$. Similarly the sheet data for normal coordinate neighborhoods containing an intersection point of a 2D Morse path and Fold surface or an intersection point of a Cusp path and Fold surface are equally easy to derive. We leave the details to the reader. The sheet data for 2D Morse Relation singularities is precisely the same as for a fold singularity, and likewise the sheet data for a Cusp Flip singularity is precisely the same as for a Cusp singularity.[1]

The remaining singularity, the swallowtail singularity, deserves more attention. The sheet data is determined by restricting attention to the graphic of Figure 2.20, which exists in a small neighborhood of the swallowtail singularity. There are three distinct regions, and so a priori there exist three sheets S, S', S''. There are also five fold arcs which relate these regions via inclusions. Three of these are drawn in Figure 2.20, and are labeled i_o, i_1, and i_2. Be virtue of the fold crossing singularity, we see that $i_0(S) \cap i_2(S) = S''$, so that this set and the remaining inclusions are already determined by the three maps $i_0, i_1, i_2 : S \to S'$. By virtue of the cusp singularities, we see that the pairs (i_0, i_1) and (i_1, i_2) form cusp sheet data. Finally, by examining the geometry of the swallowtail singularity we see that $i_0(S) \cap i_2(S) = i_0(S) \cap i_1(S) \cap i_2(S)$. These conditions are equivalent to the following three conditions on the triple of maps $(i_0, i_1, i_2) : S \to S'$:

1. $i_0(S) \cap i_2(S) = i_0(S) \cap i_1(S) \cap i_2(S)$,
2. (i_0, i_1) is valid cusp sheet data,
3. (i_1, i_2) is valid cusp sheet data.

We call such a triple *swallowtail sheet data*, $(S \Rrightarrow S')$.

Given the sheet data listed in Table 2.9, we can construct over each open set of our cover a standard elementary piece. We need to understand how to glue these together, and this requires additional data associated to the intersections of the covering sets. The covering dimension of $\mathbb{R}^2 \times I$ is three, so we may assume our cover has at most 4-fold intersections.

[1] Note that the sheet data only depends on the class of the singularity in the multi-jet Thom-Boardman classification [GG73].

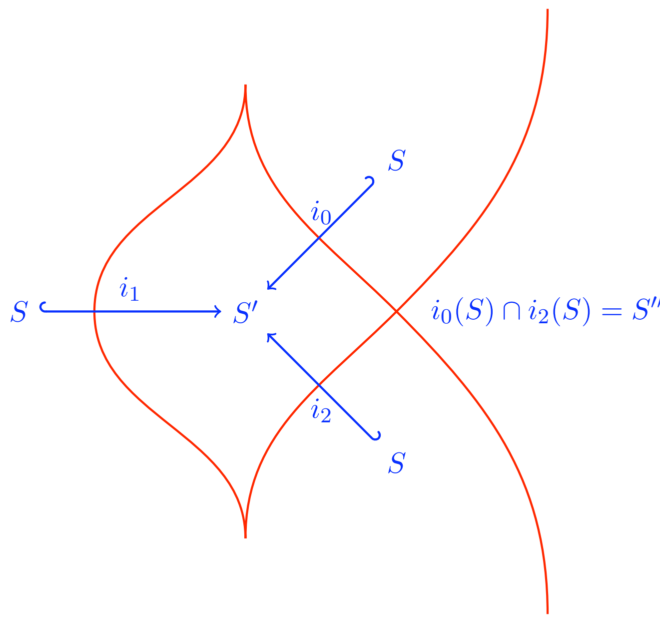


Figure 2.20: Swallowtail Sheet Data

Moreover the triple intersections come in two types: those with no singularities and those with a single fold surface. The double intersections come in five kinds. There may be no critical points, a single fold surface, a single fold crossing curve with its accompanying fold surfaces, a single 2D Morse curve and its accompanying fold surfaces, or a single cusp curve and its accompanying surfaces. Determining what this gluing data amounts to in terms of the sheet data is again straightforward. The only non-trivial case are the intersections of the cover which contain curves of cusp singularities. Suppose that we are considering such an open set of our cover, and suppose that we are given two sets of sheet data $(i_0, i_1) : S \rightrightarrows S'$ and $(\bar{i}_0, \bar{i}_1) : \overline{S} \rightrightarrows \overline{S}'$. Thus there exist two elementary pieces over this open set. The isomorphisms of these elementary pieces, relative to the map to $\mathbb{R}^2 \times I$, are given precisely by the pairs of maps $(f, f') : (S, S') \to \overline{S}, \overline{S}')$ such that the following pair of diagrams commutes,

$$\begin{tikzcd} S \arrow[r, "i_0"] \arrow[d, "f"'] & S' \arrow[d, "f'"] \\ \overline{S} \arrow[r, "\bar{i}_0"'] & \overline{S}' \end{tikzcd} \qquad \begin{tikzcd} S \arrow[r, "i_1"] \arrow[d, "f"'] & S' \arrow[d, "f'"] \\ \overline{S} \arrow[r, "\bar{i}_1"'] & \overline{S}' \end{tikzcd}$$

We call such a pair (f, f') *cusp gluing data*. The gluing data for all the double intersections is listed in Table 2.10

Table 2.10: Spacial Gluing Data

Normal Coordinate Intersection	Gluing Data
No Crit. Point	$S \cong \overline{S}$, set isom.
Fold	$(S, T) \cong (\overline{S}, \overline{T})$, pair of set isom.
2D Morse	$(S, T) \cong (\overline{S}, \overline{T})$, pair of set isom.
Fold Crossing	$(S, T_0, T_1) \cong (\overline{S}, \overline{T}_0, \overline{T}_1)$, triple of set isom.
Cusp	$(f, f') : (S, S') \to \overline{S}, \overline{S}')$, cusp gluing data

In order for the gluing to be well defined, on triple intersection the gluing data must satisfy the obvious cocycle conditions. On 4-fold intersections there are no additional conditions. This leads to the following definition.

Definition 2.5.4. A *spacial diagram* consists of a 3-dimensional graphic (in the sense of Definition 2.4.6) together with a covering by open sets, such that

1. Each open set in the covering is diffeomorphic to $\mathbb{R}^3$,
2. Each set in the covering either
   (a) contains no critical points
   (b) contains a single portion of a fold surface, and no other critical points,
   (c) contains two portions of fold surfaces, and a single crossing point curve,
   (d) contains two portions of fold surfaces, and a single crossing point curve with a single inversion point,
   (e) contains three portions of fold surface, three fold crossing curves, and a single fold triple crossing point,
   (f) contains an portion of a single 2D Morse curve, and only the accompanying two fold surfaces,

(g) contains an portion of a single 2D Morse curve, the two accompanying fold surfaces, a single additional portion of a fold surface, a fold crossing curve and a single fold-2D Morse intersection point,

(h) contains an portion of a single cusp curve, and only the accompanying two fold surfaces,

(i) contains an portion of a single cusp curve, the two accompanying fold surfaces, a single additional portion of a fold surface, a fold crossing curve and a single fold-cusp intersection point,

(j) contains a single 2D Morse relation point, and the accompanying 2D Morse curves and fold surfaces,

(k) contains a single cusp inversion point and the accompanying cusp curves and fold surfaces,

(l) contains a single cups flip point and the accompanying cusp curves, 2D Morse curves, and fold surfaces, or

(m) contains a single swallowtail point and the accompanying cusp curves, fold crossing curve, and fold surfaces.

In addition, each open set in the covering is equipped with appropriate sheet data as in Table 2.9. Furthermore we require that all double triple and quadruple intersections are connected, and that there are no 5-fold intersections. We also require that the double intersections are only of kind listed in Table 2.10, and that the double intersections are equipped with gluing data as per Table 2.10. On triple intersections, which we require to contain either no critical points or a single portion of a fold surface, we additionally require that the gluing data on the adjacent double intersections satisfy the cocycle condition. Finally, we require that the horizontal slices, $(\mathbb{R}^2 \times \{t\}) \cap U$, of each open set in the cover is a connected open set. ◇

A spacial diagram can be restricted to either of its two boundaries, and the last condition ensures that this yields a planar diagram. This leads to a natural notion of equivalence between planar diagrams.

Definition 2.5.5. Two planar diagrams are *equivalent* if there exists a spacial diagram whose restriction to the boundary agrees with the pair of planar diagrams. ◇

The notion of equivalence of planar diagrams is clearly reflexive and is symmetric by definition. It is transitive since we may concatenate spacial diagrams[2], and thus forms an equivalence relation. The detailed analysis that we have preformed on spacial diagrams no pays off. We can understand very precisely when two planar diagrams are equivalent.

Theorem 2.5.6. *Two planar diagrams are equivalent if and only if they can be related by a finite sequence of the following local moves:*

1. *Isotopy.*
2. *The changes in graphic depicted in Tables 2.11 and 2.12. These graphics must be labeled with appropriate indices (the possible number of such indices are listed in these tables), and must also be labeled with appropriate sheet data as in Table 2.9.*
3. *The changes in sheet data on individual open covering sets depicted in Figure 2.21, whenever there is a corresponding gluing datum, as in Table 2.10. Here each graphic is labeled with the appropriate index. Moreover, these local moves are required to satisfy the conditions depicted in Figures 2.22 and 2.23 on all double intersections. The obvious rotations of the moves in Figure 2.21 are also allowed.*
4. *The 1-3 Local Move depicted in Figure 2.24, in which one triple intersection region is exchanged with three.*
5. *The 2-2 Local Move depicted in Figure 2.25, in which two triple intersections are exchanged for a different pair of triple intersections.*
6. *The Fold-1 Local Move depicted in Figure 2.26 in which a fold line crosses a triple intersection point.*

Proof. Each of these moves corresponds to a simple elementary spacial diagram (see the remarks below), and so, by concatenating these elementary spacial diagrams, it is clear that if two planar diagrams can be related by a sequence of these moves, then they are equivalent.

Conversely, suppose that two planar diagrams, P_0 and P_1, are equivalent. Thus there exists a spacial diagram realizing this equivalence. If we isotope the spacial diagram,

[2]To make the glued spacial diagram smooth, we may need to adjust the original spacial diagrams slightly. This is not a serious issue since we only need to show the relation is transitive.

Table 2.11: Some Local Moves for Equivalent Planar Diagrams

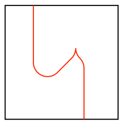
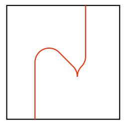
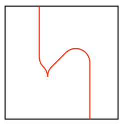
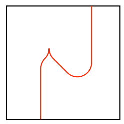
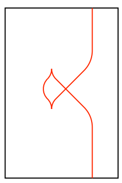
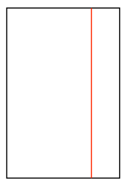
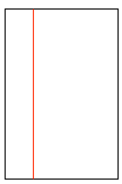
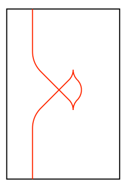
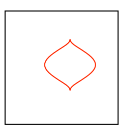
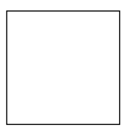
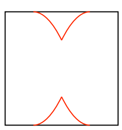
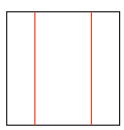
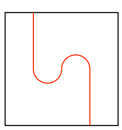
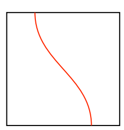
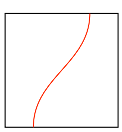
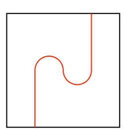
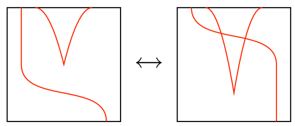
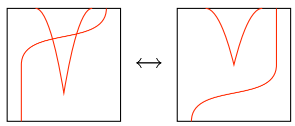
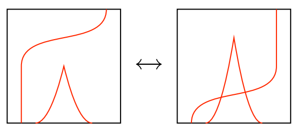
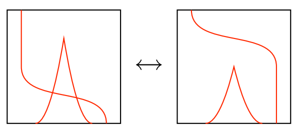
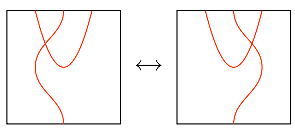
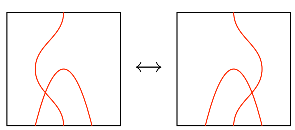
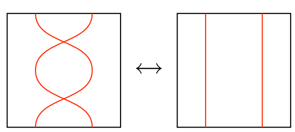
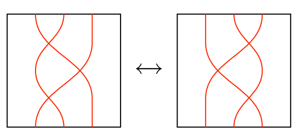
Image of the Graphic	Number of Indices
 ↔ 	1 Index
 ↔ 	1 Index
 ↔ 	1 Index
 ↔ 	1 Index
 ↔ 	1 Index
 ↔ 	1 Index
 ↔ 	2 Indices
 ↔ 	2 Indices

Table 2.12: Some More Local Moves for Equivalent Planar Diagrams

Image of the Graphic	Number of Indices
	2 Indices
	2 Indices
	2 Indices
	2 Indices
	8 Indices
	8 Indices
	4 Indices
	8 Indices

keeping it fixed on the boundary, then we achieve a new spacial diagram which still realizes the equivalence of P_0 and P_1. The results of this chapter ensure that we may isotope the spacial diagram so that it is a finite concatenation of the listed elementary spacial diagrams. □

The organization of these local moves is as follows. The isotopy moves (1) come from spacial diagrams which contain just paths of cusps, paths of 2D Morse, fold surfaces, and paths of fold crossings. The local moves in (2) occur in a single open covering set. They come from the two Tables 2.11 and 2.12, which correspond to the codimension 3 isolated singularities and the codimension 3 isolated multijet singularities. The local moves in (3) correspond to intersections of two open covering sets in the spacial diagram, and hence to the gluing data of Table 2.10. They must still satisfy a cocycle condition on triple intersections and this is precisely the condition depicted in Figures 2.22 and 2.23. The 1-3 and 2-2 local moves in (4) and (5) correspond to the two possible configurations surrounding a quadruple intersection region in a spacial diagram. Finally, the local move in (5) corresponds to a triple intersection region in the spacial diagram which contains a fold surface.

Theorem 2.5.7 (Planar Decomposition Theorem)**.** *Throughout, Σ will denote a compact surface.*

1. *A generic map $\Sigma \to \mathbb{R}^2$, together with a choice of normal coordinate neighborhoods gives rise to a planar diagram.*

2. *Any surface admits a generic map $\Sigma \to \mathbb{R}^2$, together with a choice of normal coordinate neighborhoods.*

3. *To each planar diagram, there exists a surface Σ equipped with a generic map $\Sigma \to \mathbb{R}^2$ and local coordinate neighborhoods, inducing the original planar diagram.*

4. *Given a surface, Σ, any two choices of generic maps $\Sigma \to \mathbb{R}^2$, together with choices of normal coordinate neighborhoods, induce equivalent planar diagrams.*

5. *Let Σ and Σ' be two surfaces, equipped with generic maps $\Sigma, \Sigma' \to \mathbb{R}^2$ and choices of normal coordinate neighborhoods, such that the induced planar diagrams are equivalent. Then $\Sigma \cong \Sigma'$.*

6. *Diffeomorphic surfaces induce the same equivalence class of planar diagrams.*

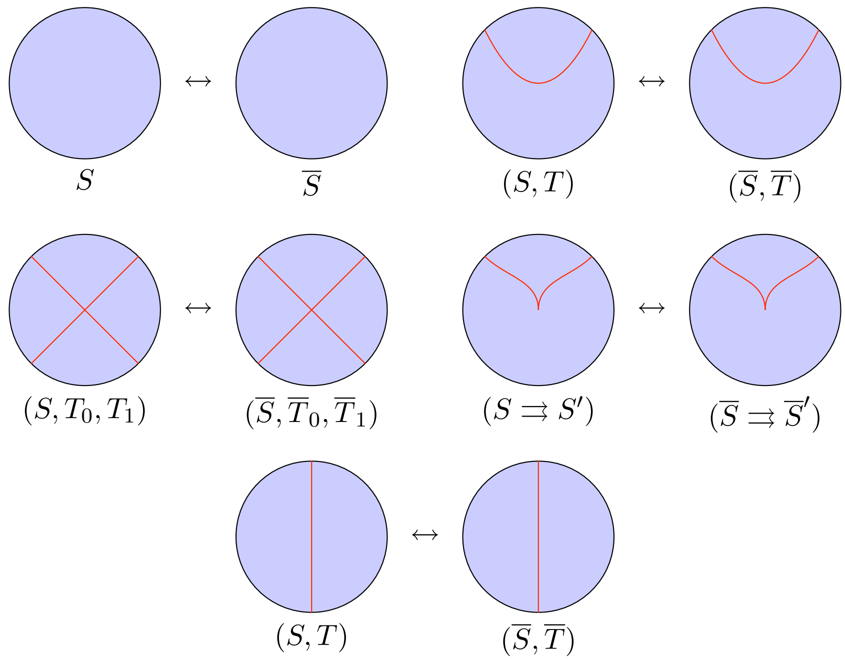
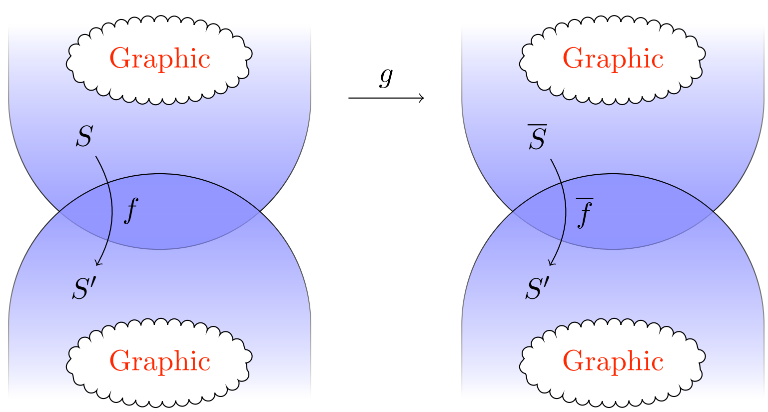
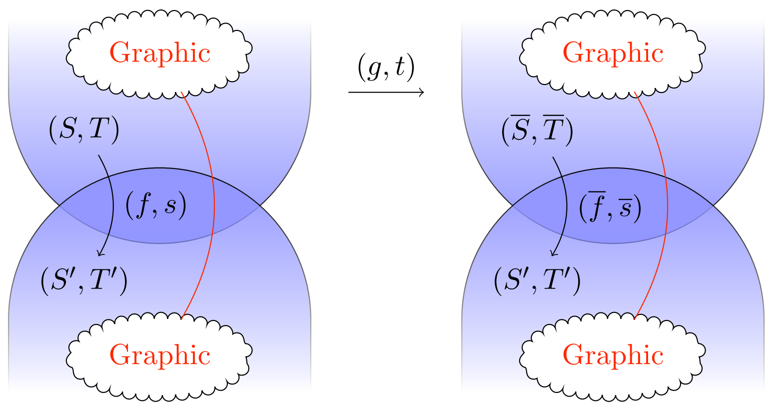


Figure 2.21: Some Local Moves for Planar Diagrams



$$f = \overline{f} \circ g$$

Figure 2.22: Some Relations for Local Moves for Planar Diagrams



$$f = \overline{f} \circ g \quad \text{and} \quad s = \overline{s} \circ t$$

Figure 2.23: Some Relations for Local Moves for Planar Diagrams

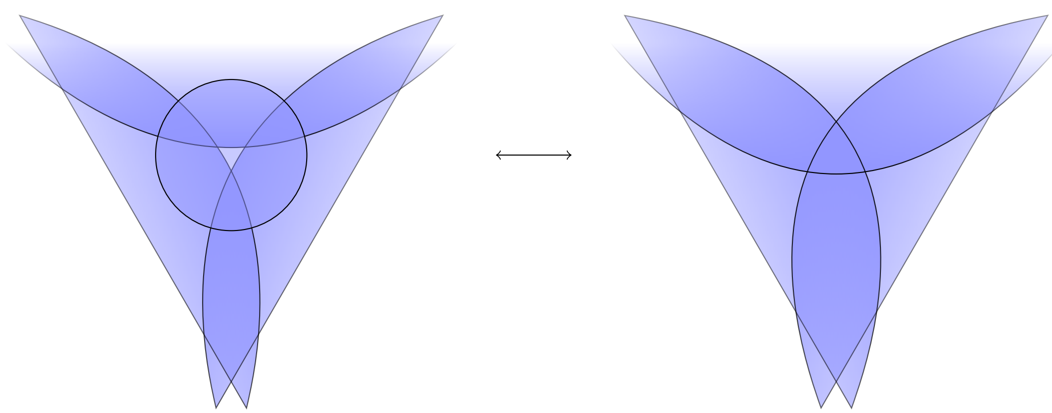


Figure 2.24: 1-3 Local Move for Planar Diagrams

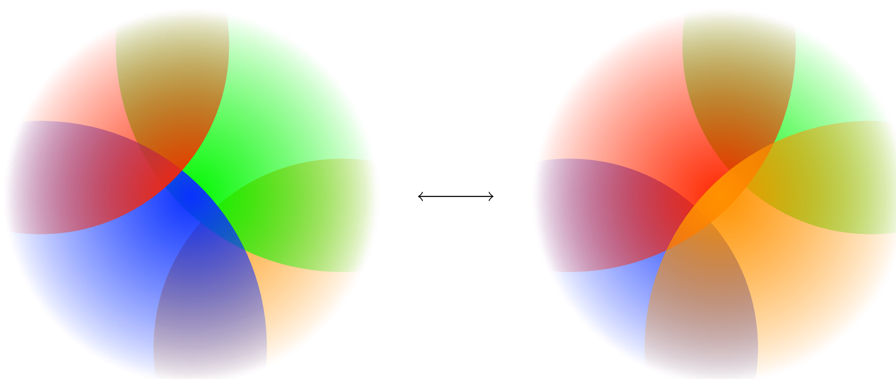


Figure 2.25: 2-2 Local Move for Planar Diagrams

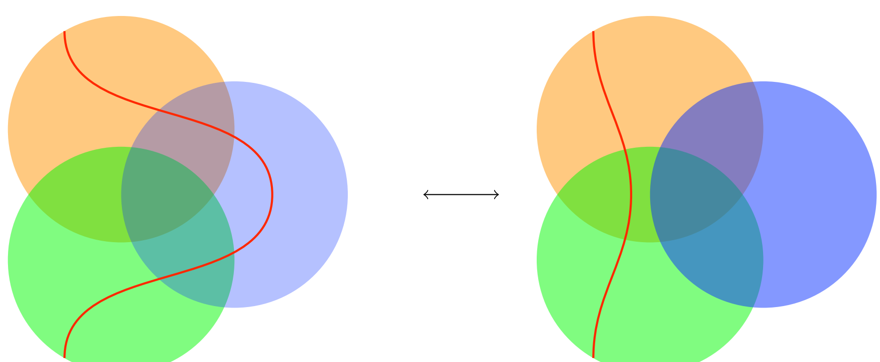


Figure 2.26: Fold-1 Local Move for Planar Diagrams

In particular isomorphism classes of surfaces are in bijection with equivalence classes of planar diagrams.

Proof. Parts (1) and (2) are merely summarizing the results of the previous sections of this chapter. The surface in part (3) is obtained by gluing together standard elementary surfaces according to the prescription given by the planar diagram. Part (4) follows because any two such generic maps, with coordinate neighborhoods can be extended to a generic map $\Sigma \times I \to \mathbb{R}^2 \times I$, with coordinate neighborhoods. This induces a spacial diagram realizing the equivalence of the original planar diagrams. Similarly, for part (5), the equivalence of planar diagrams is realized by a particular spacial diagram. The spacial diagram gives a prescription for construction a surface Σ'' and a map $\Sigma'' \times I \to \mathbb{R}^2 \times I$ inducing the spacial diagram. $\Sigma'' \times I$ is constructed by gluing the elementary pieces together, and in partiucular we have $\Sigma \cong \Sigma'' \cong \Sigma'$. Finally part (6) follows, since we use the diffeomorphism to ensure that Σ and Σ' induce the *same* planar diagram. $\square$

Chapter 3

Symmetric Monoidal Bicategories

In this chapter we introduce symmetric monoidal bicategories and prove several general theorems about them. In the first section we review the history of symmetric monoidal bicategories, explaining some of what has and what has not been done previously in the literature. In particular the definition of symmetric monoidal bicategory is scattered in several pieces throughout the literature, and there are a few minor gaps.

In Section 3.2 we collect together these result and present the fully-weak definition of symmetric monoidal bicategories, as well as the accompanying definitions of symmetric monoidal homomorphism, symmetric monoidal transformation, symmetric monoidal modification, and their various compositions. In Section 3.3 we introduce the operation of symmetric monoidal whiskering. To our knowledge this has never appeared in the literature, but an analogous operation for (non-symmetric) monoidal bicategories can be extracted from [GPS95]. This is the key operation which allows one to construct a tricategory of symmetric monoidal bicategories (in the sense of [GPS95]). See also Definition B.1.13 in the appendix for a lower categorical version of this operation.

In Section 3.4 we prove one of two key theorems about symmetric monoidal bicategories, which we dub "Whiteheads theorem for symmetric monoidal bicategories". Whitehead's theorem in topology provides a recognition principle for when a map of (nice) topological spaces is a homotopy equivalence. Theorem 3.4.10 provides a similar recognition principle for determining when a symmetric monoidal homomorphism is an equivalence of symmetric monoidal bicategories. This theorem is surely "known to experts", but as neither its proof nor its statement exists in the literature we provide a complete proof here. To reduce the amount of coherence data one needs to check, we break the proof of Theo-

rem 3.4.10 into a sequence of six lemmas, and introduce the notion of skeletal symmetric monoidal bicategories, mimicking the well known definition of skeletal categories.

In Section 3.5 we prove our other key theorem about symmetric monoidal bicategories. Here we introduce a precise definition of generating data and prove the existence of a free symmetric monoidal bicategory generated by such data. This bicategory satisfies a universal property. The bicategory of symmetric monoidal homomorphisms out of this free symmetric monoidal bicategory is equivalent to the bicategory given by specifying the value of the homomorphism on each generator. The precise details involve a lengthly inductive definition. In Section 3.6 we build on this by introducing the notion of relations for a symmetric monoidal bicategory. Together these results give a precise meaning to the concepts of generators and relations for a symmetric monoidal bicategories.

3.1 Previous Work

The mathematical axiomatization of symmetric monoidal bicategories goes back to the work of Kapranov and Voevodsky [KV94a, KV94b] on braided monoidal 2-categories. They proposed a definition of braiding for strictly associative monoidal 2-categories. There was a minor omission in this original definition of braided (strict) monoidal 2-category, which was repaired in the work of Baez and Neuchl [BN96], where the definition is also simplified and put into a more conceptual context.

It was further clarified in the work of Day and Street [DS97]. They explain how the categorification of monoidal, braided monoidal, and symmetric monoidal categories gains an additional layer. There are monoid 2-categories, braided monoidal 2-categories, *sylleptic* monoidal 2-cateogries and, finally, symmetric monoidal 2-categories. Just as symmetric monoidal categories are braided monoidal cateogries which satisfy additional axioms, symmetric monoidal bicategories are sylleptic monoidal bicategories satisfying additional axioms. This is in contrast to the relationship between braided and sylleptic monoidal bicategories: a sylleptic monoidal bicategory not only satisfies additional axioms compared to a braided monoidal bicategory, but it is equipped with additional structure, a *syllepsis*. Again, all this is carried out using a partially strict notion of monoidal 2-category called a *Gray monoid*. The authors justify this by invoking the coherence theorem of Gordon, Powers and Street.

Gordon, Powers and Street, in [GPS95], introduced what is essentially a fully weak

notion of tricategory and accompanying notions of trihomomorphism, tritransformation, and higher morphisms. Mirroring the well know definition of a (weak) monoidal category as a bicategory with one object, they define a monoidal bicategory to be a tricategory with one object. This is the most common form encountered in examples, and it is essentially the definition we use below. These authors also prove a coherence theorem, which ensures that any such monoidal bicategory is equivalent to a Gray monoid.

Day and Street consequently defined braided, sylleptic and symmetric monoidal bicategories only in the context of Gray monoids. To quote them,

> Examples naturally occur as monoidal bicategories rather than Gray monoids. However, the coherence theorem of [GPS95] allows us to transfer our definitions and results. ...

This transference was carried out explicitly for braided and sylleptic monoidal bicategories in part of the thesis work of P. McCrudden and occurs in the appendices of [McC00]. The symmetric monoidal bicategories of primary interest to the current work are, naturally, of this fully weak kind.

There are two remaining gaps, which are relatively straightforward to fill, but which we were unable to find in the literature. First, the axiom that a sylleptic monoidal bicategory must satisfy in order to be symmetric must be transfered from the partially strict version presented in [DS97] to the fully weak context. The simplicity of this axiom makes this completely straightforward. Second, while monoidal bicategories can be defined as tricategories with a single object this does not give the correct notion of monoidal transformation and monoidal modification. Indeed, if one were to adopt these as the correct notion of higher morphisms, then there would be an additional categorical layer: permutations between the modifications, as given in [GPS95]. This situation mirrors that for monoidal categories.

Monoidal categories can be defined as single object bicategories and homomorphisms between these result in monoidal functors. However transformations between these homomorphism are not the same as monoidal natural transformations between monoidal functors, and the modifications provide an additional categorical layer not usually discussed in the context of monoidal categories.

These two points of view can be reconciled by observing that the single object bicategory corresponding to a monoidal category is canonically *pointed*, and this should be considered as part of the structure. Homomorphisms between single object bicategories are

automatically pointed homomorphisms, as expected. However, pointed transformations precisely reproduce monoidal natural transformations, and moreover all pointed modifications between single object bicategories are necessarily trivial. In this way monoidal categories form a precise subset of the theory of bicategories, see Appendix B.3.

A similar discussion applies to monoidal bicategories, which should be considered as canonically pointed single object tricategories. Again all homomorphisms between these are automatically pointed and there are no non-identity pointed permutations. We take as a definition that monoidal transformations and monoidal modifications should not be given as those in [GPS95], but rather by their pointed analogs. We will only be considering pointed homomorphism, transformations and modifications which occur between single object tricategories and this considerably simplifies the relevant diagrams presented in [GPS95]. We provide these simplified diagrams below. Additionally, the following tables should help in the translation between the various structures defined in the aforementioned works.

Table 3.1: Translating Definitions of Monoidal Bicategories

Author	Monoidal ($\clubsuit$)	Braided ($\diamondsuit$)	Sylleptic ($\heartsuit$)
BN	$(\mathcal{C}, \otimes, I, id, id, id, id, id, id, id)$	$(R, \tilde{R}^{-1}_{(-\mid -,-)}, \tilde{R}^{-1}_{(-,-\mid -)})$	n/a
DS	$(\mathcal{M}, \otimes, I, id, id, id, id, id, id, id)$	$(\rho, \bar{\omega}^{-1}_{--I-}, \bar{\omega}^{-1}_{-I--})$	v
M	$(\mathcal{K}, \otimes, I, \mathsf{a}, \mathsf{l}, \mathsf{r}, \pi, \nu, \lambda, \rho)$	(ρ, R, S)	v
GPS	$(T, I, \otimes, \mathfrak{a}, \mathfrak{l}, \mathfrak{r}, \pi, \mu, \lambda, \rho)$	n/a	n/a
Here	$(\mathsf{M}, \otimes, 1, \alpha, \ell, r, \pi, \mu, \lambda, \rho)$	(β, R, S)	σ

Table 3.2: Translating Definitions of Homomorphisms of Monoidal Bicategories

Author	Monoidal ($\clubsuit$)	Braided ($\diamondsuit$)
BN	–	–
DS	$(T, \chi, \iota, \omega, \kappa, \xi)$	u
M	$(T, \chi, \iota, \omega, \gamma, \delta)$	u
GPS	$(H, \chi, \iota, \omega, \gamma, \delta)$	n/a
Here	$(H, \chi, \iota, \omega, \gamma, \delta)$	u

In what follows, we will only need to consider symmetric monoidal bicategories, and not the related notions of monoidal, braided monoidal, and sylleptic monoidal. However

Table 3.3: Translating Definitions of Transformations and Modifications of Monoidal Bicategories

Authors	BN	DS	M	GPS	Here
Monoidal Transformation (♣)	–	$(\theta, \theta_2, \theta_0)$	(θ, Π, M)	(θ, Π, M)	(θ, Π, M)
Monoidal Modification (♣)	–	s	s	m	m

these related notions might be of interest to the reader, and so we have grouped the various data and axioms accordingly. We have also given each a designated symbol. Those axioms and data which are designated with the symbol ♣ correspond to the definition of monoidal bicategories. Those designated with the symbol ♢ correspond to braided monoidal bicategories, and similarly ♡ corresponds to sylleptic and ♠ to symmetric monoidal bicategories.

3.2 Symmetric Monoidal Bicategories

Definition 3.2.1. A *symmetric monoidal bicategory* consists of a bicategory M together with the following data:

♣ a distinguished object $1 \in \mathsf{M}$,

♣ a homomorphism

$$\otimes = (\otimes, \phi^{\otimes}_{(f,f'),(g,g')}, \phi^{\otimes}_{(a,a')}) : \mathsf{M} \times \mathsf{M} \to \mathsf{M}$$

transformations:

$$\clubsuit \begin{cases} \alpha = (\alpha_{abc}, \alpha_{fgh}) : (a \otimes b) \otimes c \to a \otimes (b \otimes c) \\ \ell = (\ell_a, \ell_f) : 1 \otimes a \to a \\ r = (r_a, r_f) : a \to a \otimes 1 \end{cases}$$

$$\diamondsuit \begin{cases} \beta = (\beta_{ab}, \beta_{fg}) : a \otimes b \to b \otimes a \end{cases}$$

which are adjoint equivalence transformations. We also choose inverses α^*, ℓ^*, r^* and β^* and their associated adjunction data, which we will not name.

♣ invertible modifications:

$$
\begin{tikzcd}
& (a\otimes b)\otimes(c\otimes d) \arrow[dr, "\alpha"] & \\
((a\otimes b)\otimes c)\otimes d \arrow[ur, "\alpha"] \arrow[d, "\alpha\otimes I"'] & \Uparrow \pi & a\otimes(b\otimes(c\otimes d)) \\
(a\otimes(b\otimes c))\otimes d \arrow[rr, bend right, "\alpha"'] & & a\otimes((b\otimes c)\otimes d) \arrow[u, "I\otimes\alpha"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
& (a\otimes 1)\otimes b \arrow[r, "\alpha"] & a\otimes(1\otimes b) \arrow[dr, "I\otimes\ell"] & \\
a\otimes b \arrow[ur, "r\otimes I"] \arrow[rrr, bend right=15, "I_{a\otimes b}"'] \arrow[rrr, phantom, bend left=20, "\Downarrow\mu"] & & & a\otimes b
\end{tikzcd}
$$

$$
\begin{tikzcd}
(1\otimes a)\otimes b \arrow[rr, "\ell\otimes I"] \arrow[dr, bend right, "\alpha"'] & \Downarrow\lambda & a\otimes b \\
& 1\otimes(a\otimes b) \arrow[ur, bend right, "\ell"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
a\otimes b \arrow[rr, "I\otimes r"] \arrow[dr, bend right, "r"'] & \Downarrow\rho & a\otimes(b\otimes 1) \\
& (a\otimes b)\otimes 1 \arrow[ur, bend right, "\alpha"'] &
\end{tikzcd}
$$

♢ invertible modifications:

$$
\begin{tikzcd}
& a\otimes(b\otimes c) \arrow[r, "\beta"] & (b\otimes c)\otimes a \arrow[dr, "\alpha"] & \\
(a\otimes b)\otimes c \arrow[ur, "\alpha"] \arrow[dr, "\beta\otimes I"'] & & \Downarrow R & b\otimes(c\otimes a) \\
& (b\otimes a)\otimes c \arrow[r, "\alpha"'] & b\otimes(a\otimes c) \arrow[ur, "I\otimes\beta"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & (a \otimes b) \otimes c \arrow[r, "\beta"] & c \otimes (a \otimes b) \arrow[dr, "\alpha^*"] & \\
a \otimes (b \otimes c) \arrow[ur, "\alpha^*"] \arrow[dr, "I \otimes \beta"'] \arrow[rrr, phantom, "\Downarrow S"] & & & (c \otimes a) \otimes b \\
 & a \otimes (c \otimes b) \arrow[r, "\alpha^*"'] & (a \otimes c) \otimes b \arrow[ur, "\beta \otimes I"'] &
\end{tikzcd}
$$

♡ an invertible modification:

$$
\begin{tikzcd}
a \otimes b \arrow[rr, "I_{a \otimes b}"] \arrow[dr, bend right, "\beta"'] & \Downarrow \sigma & a \otimes b \\
 & b \otimes a \arrow[ur, bend right, "\beta"'] &
\end{tikzcd}
$$

such that the following axioms are satisfied:

♣ The equations (TA1), (TA2) and (TA3) of [GPS95] are satisfied, using the data $(\mathsf{M}, 1, \otimes, \alpha, \ell, r, \pi, \mu, \lambda, \rho)$.

♢ The equations (BA1), (BA2), (BA3) and (BA4) of [McC00] are satisfied, using the above data and also (β, R, S).

♡ The equations (SA1) and (SA2) of [McC00] are satisfied, using the above data and σ.

♠ Furthermore, the following equation is satisfied:

(SMA)
$$
\begin{tikzcd}
x \otimes y \arrow[r, "\beta"] \arrow[d, "\beta"'] \arrow[dr, "I"] & y \otimes x \arrow[d, "\beta"] \\
y \otimes x \arrow[r, "\beta"'] \arrow[u, phantom, "\Downarrow \sigma", near start, shift left=6ex] & x \otimes y
\end{tikzcd}
=
\begin{tikzcd}
x \otimes y \arrow[r, "\beta"] \arrow[d, "\beta"'] & y \otimes x \arrow[d, "\beta"] \\
y \otimes x \arrow[r, "\beta"'] \arrow[ur, "I"] & x \otimes y \arrow[u, phantom, "\Downarrow \sigma", near start, shift right=6ex]
\end{tikzcd}
$$

A bicategory equipped with the data ♣, ♢, ♡, but only satisfying axioms ♣, ♢ and ♡ is a *sylleptic monoidal bicategory*. A bicategory equipped with the data ♣ and ♢ and satisfying

the axioms ♣ and ♢ is a *braided monoidal bicategory*, and lastly a bicategory equipped only with data ♣ satisfying the axioms ♣ is a *monoidal bicategory*. ⋄

Definition 3.2.2. Let M and M' be two symmetric monoidal bicategories. A *symmetric monoidal homomorphism* $H : \mathsf{M} \to \mathsf{M}'$ consists of the following data:

♣ a homomorphism $H : \mathsf{M} \to \mathsf{M}'$

♣ transformations:

$$\chi = (\chi_{ab}, \chi_{fg}) : H(a) \otimes H(b) \to H(a \otimes b)$$
$$\iota : 1' \to H(1)$$

adjoint equivalence transformations χ^* and ι^*, together with their associated adjunction data, which we leave unnamed.

♣ invertible modifications:

$$\begin{tikzcd}
 & Ha \otimes (Hb \otimes Hc) \arrow[r, "I \otimes \chi"] & Ha \otimes H(b \otimes c) \arrow[dr, "\chi"] & \\
(Ha \otimes Hb) \otimes Hc \arrow[ur, "\alpha'"] \arrow[dr, "\chi \otimes I"'] & & \arrow[l, phantom, "\Downarrow \omega"] & H(a \otimes (b \otimes c)) \\
 & H(a \otimes b) \otimes Hc \arrow[r, "\chi"'] & H((a \otimes b) \otimes c) \arrow[ur, "H\alpha"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
H(1) \otimes Ha \arrow[r, "\chi"] \arrow[dr, phantom, "\Downarrow \gamma"] & H(1 \otimes a) \arrow[dr, "H\ell"] & \\
1' \otimes Ha \arrow[u, "\iota \otimes I"] \arrow[rr, bend right, "\ell'"'] & & Ha
\end{tikzcd}
\qquad
\begin{tikzcd}
Ha \otimes 1' \arrow[r, "I \otimes \iota"] \arrow[dr, phantom, "\Uparrow \delta"] & Ha \otimes H(1) \arrow[dr, "\chi"] & \\
Ha \arrow[u, "r'"] \arrow[rr, bend right, "Hr"'] & & H(a \otimes 1)
\end{tikzcd}$$

♢ an invertible modification:

$$\begin{tikzcd} & H(b\otimes a) \arrow[dr, "H\beta"] & \\ H(b)\otimes H(a) \arrow[ur, "\chi"] \arrow[dr, "\beta'"'] \arrow[rr, phantom, "\Downarrow u"] & & H(a\otimes b) \\ & H(a)\otimes H(b) \arrow[ur, "\chi"'] & \end{tikzcd}$$

such that the following axioms hold:

♣ Equations (HTA1) and (HTA2) of [GPS95] hold using the data $(H, \chi, \iota, \omega, \gamma, \delta)$.

♢ Equations (BHA1) and (BHA2) of [McC00] hold using the previous data and u.

♡, ♠ Equation (SHA1) of [McC00] holds.

If M and M are sylleptic monoidal bicategories, then the identical data and axioms define a *sylleptic monoidal homomorphism.* If M and M are braided monoidal bicategories, then the data ♣ and ♢ subject to axioms ♣ and ♢ define a *braided monoidal homomorphism.* If M and M' are merely monoidal bicategories, then the data and axioms ♣ define a *monoidal homomorphism.* ⋄

Definition 3.2.3. Let M and M' be symmetric monoidal bicategories and let $H, \bar{H} : \mathsf{M} \to \mathsf{M}'$ be two symmetric monoidal homomorphisms. Then a *symmetric monoidal transfomation* $\theta : H \to \bar{H}$ consists of:

♣ A transformation: $\theta = (\theta_a, \theta_f) : H \to \bar{H}$

♣ Modifications:

$$\begin{tikzcd} & H(a\otimes b) \arrow[dr, "\theta"] & \\ Ha\otimes Hb \arrow[ur, "\chi"] \arrow[d, "\theta\otimes I"'] & \Uparrow \Pi & \bar{H}(a\otimes b) \\ \bar{H}a\otimes Hb \arrow[rr, bend right, "I\otimes\theta"'] & & \bar{H}a\otimes\bar{H}b \arrow[u, "\bar{\chi}"'] \end{tikzcd}$$

$$
\begin{tikzcd}
1' \arrow[rr, "\bar{\iota}"] \arrow[dr, bend right, "\iota"'] & \arrow[d, phantom, "\Uparrow M"] & \bar{H}(1) \\
& H(1) \arrow[ur, bend right, "\theta"'] &
\end{tikzcd}
$$

such that the following axioms are satisfied:

♣ The following equations hold: (see below for details)

$$
\begin{aligned}
(MBTA1.a) &= (MBTA1.b),\\
(MBTA2.a) &= (MBTA2.b),\\
(MBTA3.a) &= (MBTA3.b).
\end{aligned}
$$

♢, ♡, ♠ Equation (BTA1) of [McC00] holds.

Braided monoidal transformations and *sylleptic monoidal transformations* are defined by the same data and axioms. If M and M' are merely monoidal bicategories, and H and $\bar{H}$ are monoidal homomorphisms, then the data and axioms ♣ define a *monoidal transformation*.

◇

(MBTA1.a)

$$
\begin{tikzcd}
& \bar{H}a(\bar{H}bHc) \arrow[r, "I(I\theta)"] \arrow[d, phantom, "\Downarrow (\alpha'_{11\theta})^*"] & \bar{H}a(\bar{H}b\bar{H}c) \arrow[r, "(\alpha')^*"] & (\bar{H}a\bar{H}b)\bar{H}c \arrow[dr, "\bar{\chi}I"] \arrow[d, phantom, "\Downarrow \phi^{\otimes'}"] & \\
(\bar{H}a\bar{H}b)Hc \arrow[ur, "\alpha'"] \arrow[urrr, "(II)\theta"] \arrow[dr, "\bar{\chi}I"'] & & & & \bar{H}(ab)\bar{H}(c) \arrow[dd, "\bar{\chi}"] \\
& \bar{H}(ab)Hc \arrow[urrr, "I\theta"'] & & & \\
(\bar{H}aHb)Hc \arrow[uu, "(I\theta)I"] \arrow[r, phantom, "\Rightarrow \Pi I"] & & & \Downarrow \Pi & \bar{H}((ab)c) \arrow[dd, "\bar{H}\alpha"] \\
& H(ab)Hc \arrow[uu, "\theta I"'] \arrow[dr, "\chi"] & & & \\
(HaHb)Hc \arrow[uu, "(\theta I)I"] \arrow[ur, "\chi I"] \arrow[dr, "\alpha'"'] & \Downarrow \omega & H((ab)c) \arrow[uurr, "\theta"'] \arrow[dr, "H\alpha"] & \Downarrow \theta_\alpha & \bar{H}(a(bc)) \\
& Ha(HbHc) \arrow[r, "I\chi"'] & HaH(bc) \arrow[r, "\chi"'] & H(a(bc)) \arrow[ur, "\theta"'] &
\end{tikzcd}
$$

(MBTA1.b)

$$
\begin{tikzcd}
 & \bar{H}a(\bar{H}bHc) \arrow[r, "I(I\theta)"] & \bar{H}a(\bar{H}b\bar{H}c) \arrow[rr, "(\alpha')^*"] \arrow[ddr, "I\bar{\chi}"'] \arrow[dr, phantom, "\Downarrow I\Pi"'] & & (\bar{H}a\bar{H}b)\bar{H}c \arrow[d, "\alpha'"'] \arrow[dr, "\bar{\chi}I"] \arrow[d, phantom, shift right=8ex, "\Leftarrow"] & \\
(\bar{H}a\bar{H}b)Hc \arrow[ur, "\alpha'"] \arrow[dr, phantom, "\Rightarrow \alpha'_{1\theta 1}"] & & & & \bar{H}a(\bar{H}b\bar{H}c) \arrow[dl, "I\bar{\chi}"'] \arrow[dr, phantom, "\Leftarrow \bar{\omega}"] & \bar{H}(ab)\bar{H}(c) \arrow[d, "\bar{\chi}"] \\
(\bar{H}aHb)Hc \arrow[u, "(I\theta)I"] \arrow[dr, "\alpha'"] & & & \bar{H}a\bar{H}(bc) \arrow[ddrr, "\bar{\chi}"'] & & \bar{H}((ab)c) \arrow[dd, "\bar{H}\alpha"] \\
 & \bar{H}a(HbHc) \arrow[uuu, "I(\theta I)"] \arrow[dr, "I\chi"] \arrow[l, phantom, "\Downarrow \alpha'_{\theta 11}"] & & & & \\
(HaHb)Hc \arrow[uu, "(\theta I)I"] \arrow[dr, "\alpha'"'] & & \bar{H}aH(bc) \arrow[uur, "I\theta"] \arrow[r, phantom, "\Downarrow \Pi"] \arrow[dl, phantom, "\Rightarrow \phi^{\otimes'}"] & {} & & \bar{H}(a(bc)) \\
 & Ha(HbHc) \arrow[uu, "\theta(II)"] \arrow[r, "I\chi"'] & HaH(bc) \arrow[u, "\theta I"'] \arrow[rr, "\chi"'] & & H(a(bc)) \arrow[ur, "\theta"'] &
\end{tikzcd}
$$

(MBTA2.a)

$$
\begin{tikzcd}
 & \bar{H}(1) \otimes Ha \arrow[rr, "I \otimes \theta"] \arrow[dr, phantom, "\Downarrow \Pi"] & & \bar{H}1 \otimes \bar{H}a \arrow[dr, "\bar{\chi}"] & \\
H(1) \otimes Ha \arrow[ur, "\theta \otimes I"] \arrow[rr, "\chi"] \arrow[dr, phantom, "\Rightarrow \gamma"] & & H(1 \otimes a) \arrow[rr, "\theta"] \arrow[d, "H\ell"] \arrow[drr, phantom, "\Leftarrow \theta_\ell"] & & \bar{H}(1 \otimes a) \arrow[dd, "\bar{H}\ell"] \\
 & & Ha \arrow[drr, "\theta"] \arrow[dd, phantom, "\Downarrow \ell'_\theta"] & & \\
1' \otimes Ha \arrow[uu, "\iota \otimes I"] \arrow[urr, "\ell'"'] \arrow[drr, "I \otimes \theta"'] & & & & \bar{H}a \\
 & & 1' \otimes \bar{H}a \arrow[urr, "\ell'"'] & &
\end{tikzcd}
$$

(MBTA2.b)

$$\begin{tikzcd}
 & \bar H(1)\otimes Ha \arrow[rr, "I\otimes\theta"] \arrow[ddr, phantom, "\Downarrow\phi^{\otimes}"] & & \bar H1\otimes\bar Ha \arrow[dr, "\bar\chi"] \arrow[dd, phantom, "\Downarrow\bar\gamma"] & \\
H(1)\otimes Ha \arrow[ur, "\theta\otimes I"] \arrow[r, phantom, "\Rightarrow M\otimes I"] & {} & & & \bar H(1\otimes a) \arrow[d, "\bar H\ell"] \\
1'\otimes Ha \arrow[u, "\iota\otimes I"] \arrow[uur, bend right, "\bar\iota\otimes I"'] \arrow[drr, "I\otimes\theta"'] & & & {} & \bar Ha \\
 & & 1'\otimes\bar Ha \arrow[uuur, "\bar\iota\otimes I"] \arrow[urr, "\ell'"'] & &
\end{tikzcd}$$

(MBTA3.a)

$$\begin{tikzcd}
 & \bar Ha\otimes 1' \arrow[r, "I\otimes\iota"] \arrow[dr, phantom, "\Downarrow\phi^{\otimes}"] & \bar Ha\otimes H(1) \arrow[dr, "I\otimes\theta"] & \\
\bar Ha \arrow[ur, "r'"] \arrow[r, phantom, "\Rightarrow r'_\theta"] & Ha\otimes 1' \arrow[u, "\theta\otimes I"] \arrow[r, "I\otimes\iota"'] \arrow[d, phantom, "\Downarrow\delta^{-1}"] & Ha\otimes H(1) \arrow[u, "\theta\otimes I"'] \arrow[d, "\chi"] \arrow[r, phantom, "\Downarrow\Pi"] & \bar H(a)\otimes\bar H(1)) \arrow[d, "\bar\chi"] \\
Ha \arrow[u, "\theta"] \arrow[ur, "r'"'] \arrow[rr, "Hr"] \arrow[drr, "\theta"'] & {} & H(a\otimes 1) \arrow[r, "\theta"] \arrow[d, phantom, "\Downarrow\theta_r"] & \bar H(a\otimes 1) \\
 & & \bar Ha \arrow[ur, "\bar Hr"'] &
\end{tikzcd}$$

(MBTA3.b)

$$\begin{tikzcd}
 & \bar Ha\otimes 1' \arrow[r, "I\otimes\iota"] \arrow[drr, bend right, "I\otimes\bar\iota"'] \arrow[d, phantom, "\Downarrow\bar\delta^{-1}"] & \bar Ha\otimes H(1) \arrow[dr, "I\otimes\theta"] \arrow[d, phantom, "\Downarrow I\otimes M"] & \\
\bar Ha \arrow[ur, "r'"] \arrow[drrr, "\bar Hr"] \arrow[dr, phantom, "\#"] & {} & {} & \bar H(a)\otimes\bar H(1)) \arrow[d, "\bar\chi"] \\
Ha \arrow[u, "\theta"] \arrow[drr, "\theta"'] & {} & & \bar H(a\otimes 1) \\
 & & \bar Ha \arrow[ur, "\bar Hr"'] &
\end{tikzcd}$$

Definition 3.2.4. (♣, ♢, ♡ and ♠) Let M and M' be symmetric monoidal bicategories, $H, \bar H : \mathsf{M} \to \mathsf{M}'$ be symmetric monoidal homomorphisms and $\theta, \tilde\theta : H \to \bar H$ be symmetric

monoidal transformations. Then a *symmetric monoidal modification* $m : \theta \to \tilde{\theta}$ consists of a modification $m : \theta \to \tilde{\theta}$ such that the following two equations hold:

(BMBM1)

$$
\begin{tikzcd}
Ha \otimes Hb \arrow[r, "\chi"] \arrow[d, bend right=60, "\theta\otimes\theta"'] \arrow[dr, phantom, "\Rightarrow \Pi"] & H(a\otimes b) \arrow[d, "\theta"'] \arrow[d, bend left=60, "\tilde{\theta}"] \arrow[d, phantom, shift left=4, "\Rightarrow m"] \\
\bar{H}a \otimes \bar{H}b \arrow[r, "\bar{\chi}"'] & \bar{H}(a\otimes b)
\end{tikzcd}
=
\begin{tikzcd}
Ha \otimes Hb \arrow[r, "\chi"] \arrow[d, bend right=60, "\theta\otimes\theta"'] \arrow[d, bend left=60, "\tilde{\theta}\otimes\tilde{\theta}"] \arrow[d, phantom, "\Rightarrow m\otimes m"] \arrow[dr, phantom, "\Rightarrow \tilde{\Pi}"] & H(a\otimes b) \arrow[d, bend left=60, "\tilde{\theta}"] \\
\bar{H}a \otimes \bar{H}b \arrow[r, "\bar{\chi}"'] & \bar{H}(a\otimes b)
\end{tikzcd}
$$

(BMBM2)

$$
\begin{tikzcd}
 & H(1) \arrow[d, "\theta"'] \arrow[d, bend left=60, "\tilde{\theta}"] \arrow[d, phantom, shift left=4, "\Rightarrow m"] \\
1 \arrow[ur, bend left, "\iota"] \arrow[dr, bend right, "\bar{\iota}"'] \arrow[r, phantom, "\Rightarrow M"] & {} \\
 & \bar{H}(1)
\end{tikzcd}
=
\begin{tikzcd}
 & H(1) \arrow[dd, bend left=60, "\tilde{\theta}"] \\
1 \arrow[ur, bend left, "\iota"] \arrow[dr, bend right, "\bar{\iota}"'] \arrow[r, phantom, "\Rightarrow \tilde{M}"] & {} \\
 & \bar{H}(1)
\end{tikzcd}
$$

◇

Definition 3.2.5. Let M and M' be two symmetric monoidal bicategories, let $H, \overline{H} : \mathsf{M} \to \mathsf{M}'$ be symmetric monoidal homomorphisms, let $\theta, \tilde{\theta}, \tilde{\tilde{\theta}} : H \to \overline{H}$ be symmetric monoidal transformations, and let $m : \theta \to \tilde{\theta}$ and $\tilde{m} : \tilde{\theta} \to \tilde{\tilde{\theta}}$ be symmetric monoidal modifications. Then the modification $\tilde{m} \circ m : \theta \to \tilde{\tilde{\theta}}$ is a symmetric monoidal modification called the *(vertical) composition* of the symmetric monoidal modifications $m, \tilde{m}$. ◇

Definition 3.2.6. Let M and M' be two symmetric monoidal bicategories, let $H, \overline{H}, \overline{\overline{H}} : \mathsf{M} \to \mathsf{M}'$ be symmetric monoidal homomorphisms, and let $\theta : H \to \overline{H}$, $\tilde{\theta} : \overline{H} \to \overline{\overline{H}}$ be symmetric monoidal transformations. Then the *composition* of θ and $\tilde{\theta}$ is the symmetric monoidal transformation given by the transformation $\tilde{\theta} \circ \theta$ and the modifications whose components are given by the following pasting diagrams:

$$
\begin{tikzcd}
 & & H(1 \otimes a) \arrow[dr, "\theta"] & & & \\
H(1) \otimes Ha \arrow[urr, "\chi"] \arrow[ddr, "\theta \otimes I"] \arrow[dddrr, bend right, "(\tilde{\theta} \circ \theta) \otimes I"'] & & \Uparrow \Pi & \bar{H}(1 \otimes a) \arrow[drr, "\tilde{\theta}"] & & \\
 & & & & \Uparrow \tilde{\Pi} & \overline{\overline{H}}(1 \otimes a \\
 & \bar{H}(1) \otimes Ha \arrow[rr, "I \otimes \theta"] \arrow[dr, "\tilde{\theta} \otimes I"] \arrow[l, phantom, "\Rightarrow"] & \Uparrow & \bar{H}(1) \otimes \bar{H}a \arrow[uu, "\bar{\chi}"] \arrow[dr, "\tilde{\theta} \otimes I"] & & \\
 & & \overline{\overline{H}}(1) \otimes H(a) \arrow[rr, "I \otimes \theta"] \arrow[rrr, bend right, "I \otimes (\tilde{\theta} \circ \theta)"'] & & \overline{\overline{H}}(1) \otimes \overline{H}(a) \arrow[r, "I \otimes \tilde{\theta}"] \arrow[d, phantom, "\Uparrow"] & \overline{\overline{H}}(1) \otimes \overline{\overline{H}}(a) \arrow[uu, "\overline{\overline{\chi}}"'] \\
 & & & & {} &
\end{tikzcd}
=: \Pi^{\tilde{\theta}\theta}
$$

$$
M^{\tilde{\theta}\theta} :=
\begin{tikzcd}
1' \arrow[rr, "\bar{\bar{\iota}}"] \arrow[dr, "\bar{\iota}"'] \arrow[dd, "\iota"'] & \Uparrow \tilde{M} & \overline{\overline{H}}(1) \\
\Rightarrow M & \overline{H}(1) \arrow[ur, "\tilde{\theta}"'] & \\
H(1) \arrow[ur, "\theta"'] & &
\end{tikzcd}
$$

$\diamond$

Definition 3.2.7. Let M and M' be two symmetric monoidal bicategories, let $H, \overline{H}, \overline{\overline{H}} : \mathsf{M} \to \mathsf{M}'$ be symmetric monoidal homomorphisms, let $\theta, \xi : H \to \overline{H}$, $\tilde{\theta}, \tilde{\xi} : \overline{H} \to \overline{\overline{H}}$ be symmetric monoidal transformations and let $m : \theta \to \xi$ and $\tilde{m} : \tilde{\theta} \to \tilde{\xi}$ be symmetric monoidal modifications. Then the *horizontal composition* of m and $\tilde{m}$ is the symmetric monoidal modification $\tilde{m} * m$. $\diamond$

Together with these compositions and the obvious associators and unitors, we have a bicategory $\mathsf{SymBicat}(\mathsf{M}, \mathsf{M}')$ whose objects are the symmetric monoidal homomorphisms, the 1-morphism are the symmetric monoidal transformations, and whose 2-morphisms are the symmetric monoidal modifications.

3.3 Symmetric Monoidal Whiskering

Definition 3.3.1. Let M, M' and M'' be symmetric monoidal bicategories. Let $H : \mathsf{M} \to \mathsf{M}'$ and $\overline{H} : \mathsf{M}' \to \mathsf{M}''$ be symmetric monoidal homomorphisms. Then the *composition* of H and $\overline{H}$ is the symmetric monoidal homomorphism given by the homomorphism $\overline{H}H : \mathsf{M} \to \mathsf{M}''$, the following transformations:

$$\chi^{\overline{H}H} : \overline{H}H(a) \otimes'' \overline{H}H(b) \xrightarrow{\overline{\chi}} \overline{H}(Ha \otimes' Hb) \xrightarrow{\overline{H}\chi} \overline{H}H(a \otimes b)$$
$$\iota^{\overline{H}H} : 1'' \xrightarrow{\overline{\iota}} \overline{H}(1') \xrightarrow{\overline{H}\iota} \overline{H}H(1)$$

and the modifications whose components are given by the following pasting diagrams. Here the unlabeled 2-morphisms are the components of the transformations $\chi, \iota, \overline{\chi}$ and $\overline{\iota}$ and the canonical coherence 2-morphisms from M''.

$$\omega^{\overline{H}H} :=$$

$$\begin{tikzcd}
& \overline{H}Ha \otimes (\overline{H}Hb \otimes \overline{H}Hc) \arrow[rr, "I'' \otimes \overline{\chi}"] \arrow[rrrr, bend left=40, "I'' \otimes {[\overline{H}(\chi) \circ \overline{\chi}]}"] \arrow[rrrr, phantom, bend left=20, "\Downarrow"] & & \overline{H}Ha \otimes \overline{H}(Hb \otimes Hc) \arrow[rr, "I'' \otimes \overline{H}(\chi)"] \arrow[rr, bend right, "\overline{H}(I') \otimes \overline{H}(\chi)"'] \arrow[rr, phantom, "\Downarrow"] \arrow[dr, "\overline{\chi}"] & & \overline{H}Ha \otimes \overline{H}H(b \otimes c) \arrow[dr, "\overline{\chi}"] & & \\
(\overline{H}Ha \otimes \overline{H}Hb) \otimes \overline{H}Hc \arrow[ur, "\alpha''"] \arrow[dr, "\overline{\chi} \otimes I''"] \arrow[ddrr, bend right=60, "{[\overline{H}(\chi) \circ \overline{\chi}]} \otimes I''"'] \arrow[rrrr, phantom, "\Downarrow \overline{\varpi}"] & & & & \overline{H}(Ha \otimes (Hb \otimes Hc)) \arrow[rr, "\overline{H}(I' \otimes \chi)"'] \arrow[ur, phantom, "\Downarrow"] & & \overline{H}(Ha \otimes H(b \otimes c)) \arrow[dr, "\overline{H}\chi"] & \\
& \overline{H}(Ha \otimes Hb) \otimes \overline{H}Hc \arrow[rr, "\overline{\chi}"'] \arrow[dr, "(\overline{H}\chi) \otimes I''"'] \arrow[dr, bend left=60, "\overline{H}(\chi) \otimes \overline{H}(I')"] \arrow[dr, phantom, "\Leftarrow"] \arrow[l, phantom, "\Leftarrow"] & & \overline{H}((Ha \otimes Hb) \otimes Hc) \arrow[ur, "\overline{H}\alpha'"] \arrow[dr, "\overline{H}(\chi \otimes I')"] \arrow[rrrr, phantom, "\Downarrow \overline{H}\omega"] & & & & \overline{H}H(a \otimes (b \otimes c)) \\
& & \overline{H}H(a \otimes b) \otimes \overline{H}Hc \arrow[rr, "\overline{\chi}"'] \arrow[ur, phantom, "\Downarrow"] & & \overline{H}(H(a \otimes b) \otimes Hc) \arrow[rr, "\overline{H}\chi"'] & & \overline{H}H((a \otimes b) \otimes c) \arrow[ur, "\overline{H}H\alpha"'] &
\end{tikzcd}$$

$$\gamma^{\overline{H}H} := \begin{tikzcd}
\overline{H}H(1)\otimes\overline{H}H(a) \arrow[r, "\overline{\chi}"] & \overline{H}(H(1)\otimes H(a)) \arrow[r, "\overline{H}(\chi)"] & \overline{H}H(1\otimes a) \arrow[dd, bend left, "\overline{H}H(\ell)"] \\
\overline{H}(1')\otimes\overline{H}Ha \arrow[u, "\overline{H}(\iota)\otimes I''"] \arrow[u, bend right, "\overline{H}(\iota)\otimes\overline{H}(I')"'] \arrow[u, phantom, "\Rightarrow"] \arrow[r, "\overline{\chi}"'] \arrow[ur, phantom, "\Downarrow"] \arrow[dr, phantom, "\Downarrow\overline{\gamma}"] & \overline{H}(1'\otimes Ha) \arrow[u, "\overline{H}(\iota\otimes I')"'] \arrow[dr, "\overline{H}(\ell')"] \arrow[r, phantom, "\overline{H}\gamma\Rightarrow"] & \\
1''\otimes\overline{H}Ha \arrow[u, "\overline{\iota}\otimes I''"'] \arrow[uu, bend left=60, "(\overline{H}(\iota)\circ\overline{\iota})\otimes I''"] \arrow[u, phantom, "\Rightarrow", shift right=20] \arrow[rr, bend right, "\ell''"'] & & \overline{H}Ha
\end{tikzcd}$$

$$\delta^{\overline{H}H} := \begin{tikzcd}
\overline{H}Ha\otimes 1'' \arrow[r, "I''\otimes\overline{\iota}"] \arrow[rr, bend left=40, "I''\otimes(\overline{H}(\iota)\circ\overline{\iota})"] \arrow[rr, phantom, bend left=20, "\Downarrow"] \arrow[dr, phantom, "\Rightarrow\overline{\delta}"] & \overline{H}H(a)\otimes\overline{H}(1') \arrow[r, "I''\otimes\overline{H}(\iota)"] \arrow[r, bend right, "\overline{H}(I')\otimes\overline{H}(\iota)"'] \arrow[r, phantom, "\Downarrow"] \arrow[d, "\overline{\chi}"'] \arrow[dr, phantom, "\Downarrow"] & \overline{H}H(a)\otimes\overline{H}H(1) \arrow[d, "\overline{\chi}"] \\
& \overline{H}(Ha\otimes 1') \arrow[r] & \overline{H}(H(a)\otimes H(1)) \arrow[d, "\overline{H}\chi"'] \\
\overline{H}Ha \arrow[uu, "r''"] \arrow[ur] \arrow[rr, bend right, "\overline{H}Hr"'] \arrow[rr, phantom, "\Downarrow\overline{H}\delta"] & & \overline{H}H(a\otimes 1)
\end{tikzcd}$$

$$u^{\overline{H}H} := \begin{tikzcd}
& & \overline{H}H(b\otimes a) \arrow[dr, "\overline{H}H\beta"] & \\
& \overline{H}(Hb\otimes Ha) \arrow[ur, "\overline{H}\chi"] \arrow[dr, "\overline{H}\beta'"] \arrow[rr, phantom, "\Downarrow\overline{H}u"] & & \overline{H}H(a\otimes b) \\
\overline{H}H(b)\otimes\overline{H}H(a) \arrow[ur, "\overline{\chi}"] \arrow[dr, "\beta''"'] \arrow[rr, phantom, "\Downarrow\overline{u}"] & & \overline{H}(Ha\otimes Hb) \arrow[ur, "\overline{H}\chi"'] & \\
& \overline{H}H(a)\otimes\overline{H}H(b) \arrow[ur, "\overline{\chi}"'] & &
\end{tikzcd}$$

◇

Definition 3.3.2. Let $\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}$ be symmetric monoidal bicategories, $F : \mathsf{A} \to \mathsf{B}$, $G, \overline{G} : \mathsf{B} \to \mathsf{C}$, and $H : \mathsf{C} \to \mathsf{D}$ be symmetric monoidal homomorphisms, and $\theta : G \to \overline{G}$ be a symmetric monoidal transformation. Then the *whiskering* of F and θ is the symmetric monoidal homomorphism from GF to $\overline{G}F$ defined by the transformation $\theta F : GF \to \overline{G}F$ and the modifications whose components are given by the following pasting diagrams:

$$\Pi^{\theta F} := \quad \begin{tikzcd}
 & & & GF(1 \otimes a) \arrow[dr, "\theta"] \arrow[d, phantom, "\Uparrow \theta"] & \\
 & & G(F(1) \otimes F(a)) \arrow[ur, "G\chi^F"] \arrow[drr, "\theta"] & {} & \overline{G}F(1 \otimes a) \\
GF(1) \otimes GFa \arrow[urr, "\chi^G"] \arrow[dr, "\theta \otimes I"'] \arrow[rrrr, phantom, "\Uparrow \Pi"] & & & & \overline{G}((F1) \otimes F(a)) \arrow[u, "\overline{G}\chi^F"'] \\
 & \overline{G}F(1) \otimes GFa \arrow[rr, bend right, "I \otimes \theta"'] & & \overline{G}F(1) \otimes \overline{G}F(a) \arrow[ur, "\chi^{\overline{G}}"'] &
\end{tikzcd}$$

$$M^{\theta F} := \quad \begin{tikzcd}
1 \arrow[rr, "\iota^{\overline{G}}"] \arrow[dr, "\iota^G"'] & & \overline{G}(1) \arrow[r, "\overline{G}\iota^F"] & \overline{G}F(1) \\
 & G(1) \arrow[ur, bend right, "\theta"'] \arrow[dr, "G\iota^F"'] \arrow[u, phantom, "\Uparrow M"] & \arrow[ur, phantom, "\Uparrow \theta"] & \\
 & & GF(1) \arrow[uur, "\theta"'] &
\end{tikzcd}$$

The *whiskering* of θ and H is the symmetric monoidal transformation from HG to $H\overline{G}$ given by the transformation $H\theta : HG \to H\overline{G}$ and the modifications whose components are given by the following pasting diagrams:

$$M^{H\theta} := \quad \begin{tikzcd}
1 \arrow[r, "\iota^H"] & H(1) \arrow[rr, "H\iota^{\overline{G}}"] \arrow[dr, bend right, "H\iota^G"'] & & H\overline{G}(1) \\
 & & HG(1) \arrow[ur, bend right, "H\theta"'] \arrow[u, phantom, "\Uparrow H(M)"] &
\end{tikzcd}$$

$$\Pi^{H\theta} := \begin{tikzcd}[column sep=small]
 & & & HG(1 \otimes a) \arrow[drr, "H\theta"] & & \\
 & H(F(1) \otimes F(a)) \arrow[urr, "H\chi^G"] \arrow[dr, "H(\theta \otimes I)"'] \arrow[rrrr, phantom, "\Uparrow H\Pi"] & & & & H\overline{G}(1 \otimes a) \\
HG(1) \otimes HG(a) \arrow[ur, "\chi^H"] \arrow[dr, "H\theta \otimes I"'] \arrow[dr, bend left, "H\theta \otimes HI"] \arrow[rr, phantom, "\Uparrow"] & & H(\overline{G}(1) \otimes G(a)) \arrow[rr, "H(I \otimes \theta)"'] & & H(\overline{G}(1) \otimes \overline{G}(a)) \arrow[ur, "H\chi^{\overline{G}}"'] & \\
 & H\overline{G}(1) \otimes HG(a) \arrow[ur, "\chi^H"] \arrow[rr, "I \otimes H\theta"'] \arrow[rr, bend left, "HI \otimes H\theta"] \arrow[u, phantom, "\Rightarrow"] & & H\overline{G}(1) \otimes H\overline{G}(a) \arrow[ur, "\chi^H"'] \arrow[u, phantom, "\Uparrow"] \arrow[l, phantom, "\Uparrow"] & &
\end{tikzcd}$$

Here the unlabeled arrows are the structure 2-morphisms from the transformation χ^H and the homomorphism H.

If $\tilde{\theta} : G \to \overline{G}$ is another symmetric monoidal homomorphism, and $m : \theta \to \tilde{\theta}$ is a symmetric monoidal modification, then the *whiskering* of m by H and F is the symmetric monoidal modification consisting of the modification $HmF : H\theta F \to H\tilde{\theta}F$. $\diamond$

Definition 3.3.3. A symmetric monoidal homomorphism $H : \mathsf{M} \to \mathsf{M}'$ is an *equivalence* of symmetric monoidal bicategories if there exists a symmetric monoidal homomorphism $F : \mathsf{M}' \to \mathsf{M}$ such that $FH \simeq id_{\mathsf{M}}$ in the bicategory $\mathsf{SymBicat}(\mathsf{M}, \mathsf{M})$ and $HF \simeq id_{\mathsf{M}'}$ in the bicategory $\mathsf{SymBicat}(\mathsf{M}', \mathsf{M}')$. $\diamond$

Using whiskering, it is straightforward to construct the necessary compositions which realize symmetric monoidal bicategories as a tricategory. This is done exactly as it was done for ordinary bicategories. The hom bicategories are precisely the bicategories $\mathsf{SymBicat}(\mathsf{M}, \mathsf{M}')$ considered previously. Equivalence in any tricategory is defined exactly as above, and we moreover have the following result, an analog of which is true in any tricategory.

Lemma 3.3.4. *Let M and M' be symmetric monoidal bicategories, and let $H : \mathsf{M} \to \mathsf{M}'$ be a symmetric monoidal homomorphism, which is an equivalence. Then for all symmetric*

monoidal bicategories B*, the canonical homomorphisms (induced by whiskering)*

$$H^* : \mathsf{SymBicat}(\mathsf{M}', \mathsf{B}) \to \mathsf{SymBicat}(\mathsf{M}, \mathsf{B})$$
$$H_* : \mathsf{SymBicat}(\mathsf{B}, \mathsf{M}) \to \mathsf{SymBicat}(\mathsf{B}, \mathsf{M}')$$

are equivalences of bicategories.

3.4 Whitehead's Theorem for Symmetric Monoidal Bicategories

Whitehead's theorem for topological spaces states that a map between sufficiently nice[1] topological spaces is a homotopy equivalence if and only if it induces an isomorphism of all homotopy groups (at all base points). A direct analog of this is the statement that a functor between groupoids $F : G \to G'$ is an equivalence of categories if and only if it induces a bijection on the set of isomorphisms classes of objects (i.e. is an isomorphism on π_0) and for each object $x \in G$, F induces an isomorphism $F : G(x,x) \to G'(Fx, Fx)$ (i.e. is an isomorphism on $\pi_1(G, x)$ for each object $x \in G$).

A similar well known statement holds for functors between categories: a functor $F : C \to C'$ is an equivalence of categories if and only if it induces a surjection of the set of isomorphism classes of objects (i.e. F is essentially surjective) and induces a bijection $C(x,y) \to C'(Fx, Fy)$ for all pairs of objects $x, y \in C$ (i.e. F is fully-faithful).

In this section we prove an analogous statement for symmetric monoidal bicategories, which allows us to easily verify when a homomorphism between symmetric monoidal bicategories is an equivalence. This will later be used to prove that the small model of the bordism cateogry that we will construct later is equivalent to the geometric bordism bicategory.

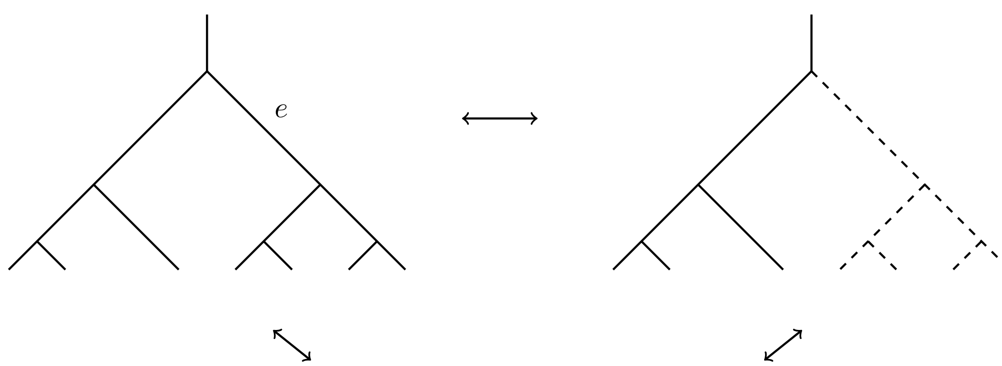
We begin with a small combinatorial digression on binary trees. Although, as stated, Lemma 3.4.3 is evidently true and it's proof relatively straightforward, we will see that it implies a sort of "coherence theorem" which will simplify later proofs and constructions. This should be compared with the construction given in [Mac71, VII.2], which is in fact more complicated then the one presented here.

Definition 3.4.1. A *binary tree* is defined recursively as follows. The symbol $(-)$ is a

[1] Here "sufficiently nice" can be taken to mean spaces with the homotopy type of a CW-complex.

binary tree. If u and v are binary trees, then $u \boxdot v = (u) \boxdot (v)$ is a binary tree. Similarly the *edges* of a binary tree are defined recursively as a set of subtrees. The edges of the binary tree $(-)$ consist of the set $\{(-)\}$. The edges of a binary tree $u \boxdot v$ consist of the disjoint union of the edges of u, the edges of v, and the set $\{u \boxdot v\}$. A *marked* binary tree (t, S) consists of a pair where t is a binary tree and S is a subset of the edges of t. $\diamond$

The definition given above is logically equivalent to the usual definition of a proper binary planar rooted tree. Every such tree has a top-most edge (the root) and so every edge e can be equivalently described by specifying the maximal subtree with e as its root, see Figure 3.1. The edges of a tree form a partially ordered set with partial order given by inclusion of subtrees. We refer to this partial order as the *height* of the edge and thus may speak of edges which have comparable height and edges which have incomparable height.



$$t = (((-) \boxdot (-)) \boxdot (-)) \boxdot \underbrace{(((-) \boxdot (-)) \boxdot ((-) \boxdot (-)))}_{e}$$

Figure 3.1: Equivalent Descriptions of Binary Trees and Edges

Definition 3.4.2. A *basic path* from the marked binary tree (t, S) to the marked binary tree (t, S') (which have the same underlying binary tree t) is an edge $e \in S$ such that $S' = S \setminus \{e\}$. A *path* is a composable chain of basic paths. $\diamond$

In other words, a path from (t, S) to (t, S') (which only exists if $S' \subset S$) is a word whose letters consist precisely of the edges $S \setminus S'$ with no omissions or repetitions. We also consider the trivial/empty word to be a path from the marked tree (t, S) to itself. A path has an evident *length*, given by the length of the chain of basic paths.

Lemma 3.4.3. *Let $\sim$ be the finest equivalence relation on the paths between marked binary trees such that the following three properties are satisfied:*

(i) For any paths $u : (t, S) \to (t, S')$, $w, \hat{w} : (t, S') \to (t, S'')$ and $v : (t, S'') \to (t, S''')$, if $w \sim \hat{w}$, then $vwu \sim v\hat{w}u$. (Compatible with composition)

(ii) Let e_0, e_1 be two edges of t such that $e_0 \leq e_1$. Let S be any subset of the edges of t, not containing e_0 or e_1. Then the following paths are equivalent:

$$(t, S \cup \{e_0, e_1\}) \stackrel{e_1}{\to} (t, S \cup \{e_0\}) \stackrel{e_0}{\to} (t, S)$$
$$(t, S \cup \{e_0, e_1\}) \stackrel{e_0}{\to} (t, S \cup \{e_1\}) \stackrel{e_1}{\to} (t, S).$$

(Edges of different heights commute)

(iii) Let e_0, e_1 be two edges of t which are incomparable. Let S be any subset of the edges of t, not containing e_0 or e_1. Then the following paths are equivalent:

$$(t, S \cup \{e_0, e_1\}) \stackrel{e_1}{\to} (t, S \cup \{e_0\}) \stackrel{e_0}{\to} (t, S)$$
$$(t, S \cup \{e_0, e_1\}) \stackrel{e_0}{\to} (t, S \cup \{e_1\}) \stackrel{e_1}{\to} (t, S).$$

(Edges of incomparable heights commute)

Let P and Q be any two paths from (t, S) to (t, S'), then $P \sim Q$.

Proof. P and Q are words on precisely the same set of letters, $S \setminus S'$. The above lemma says essentially that two such words can be rearranged to be the same word. We prove this by induction on the length of the path. Consider the following induction hypothesis which consists of the pair of statements:

(A) Let P and Q be two paths of length ℓ, starting and ending at corresponding marked binary trees. Then $P \sim Q$.

(B) Let P be a path of length ℓ and let e be a basic edge. Then $Pe \sim eP$.

The base case of our induction is for length one paths, i.e basic paths. In this case (A) is trivial since $P = Q$, and (B) follows from properties (ii) and (iii) of the equivalence relation.

For the induction step, assume that (A) and (B) hold for all paths of length ℓ or less. Let P and Q be paths of length $\ell + 1$, starting and ending at corresponding marked

binary trees. Then $P = eP'$ for a basic path e and $Q = Q_0 e Q_1$ for some paths Q_0 and Q_1. We have,

$$\begin{aligned} Q &= Q_0 e Q_1 \\ &\sim eQ_0Q_1 \quad \text{[by (B) and (i)]} \\ &\sim eP' \quad \text{[by (A) and (i)]} \\ &= P. \end{aligned}$$

This proves statement (A) for paths of length $\ell+1$. Similarly, if P is a path of length $\ell+1$, then $P = fP'$ for a basic path f and a path P' of length ℓ. If e is any basic path, then

$$\begin{aligned} Pe &= fP'e \\ &\sim feP' \quad \text{[by (B) and (i)]} \\ &\sim efP' \quad \text{[by (i), (ii) and (iii)]} \\ &= eP \end{aligned}$$

which proves statement (B) for paths of length $\ell+1$. □

We now turn our attention to symmetric monoidal bicategories. The significance of the previous lemma will become clear momentarily. The goal of this section is to prove Theorem 3.4.10, which states that a symmetric monoidal homomorphisms between symmetric monoidal bicategories is a symmetric monoidal equivalence if and only if it is an equivalence of the underlying bicategories, which in turn is true if an only if the functor is essentially surjective on objects and induces equivalence of hom categories. While it is possible to prove this theorem in a single direct calculation (by building an inverse symmetric monoidal homomorphism explicitly using the axiom of choice) the magnitude of such a calculation (for example checking this homomorphism is actually an inverse) would soon become overwhelming.

Instead we prefer to break the problem into more calculationally manageable pieces. We first introduce notions of 0- and 1-skeletal symmetric monoidal bicategories and prove that any symmetric monoidal bicategory is equivalent to one of these. This permits us to reduce the general Whitehead's theorem to the case when both bicategories are skeletal, a case which is more tractable. Recall that a category C is *skeletal* if $x \cong y$ implies $x = y$ for any objects $x, y \in C$.

Definition 3.4.4. A bicategory B is *1-skeletal* if the hom bicategories $\mathsf{B}(x, y)$ are skeletal for each pair of objects $x, y \in \mathsf{B}$. B is *0-skeletal* when it satisfies the condition: x is equivalent to y if and only if $x = y$ in B. Finally, a bicategory B is *skeletal* if it is both 0-skeletal and 1-skeletal. A symmetric monoidal bicategory is k-skeletal if it underlying bicategory is k-skeletal.[2] $\diamond$

The following lemma is well known, but we reproduce it here since we will need to use the details of its proof in what follows. It also serves as a means to illustrate how we will employ Lemma 3.4.3 in later calculations.

Lemma 3.4.5. *Every bicategory is equivalent to a 1-skeletal bicategory.*

Proof. Let B be a given bicategory. We will build a 1-skeletal bicategory B^s and an equivalence $\mathsf{B}^s \simeq \mathsf{B}$. The objects of B^s will coincide with those of B. Choose for each hom bicategory $\mathsf{B}(x, y)$ representatives for each isomorphism class of 1-morphism. Let $\mathsf{B}^s(x, y)$ be the full subcategory with these 1-morphisms as objects. $\mathsf{B}^s(x, y)$ is skeletal, and the fully-faithful inclusion functor $i : \mathsf{B}^s(x, y) \hookrightarrow \mathsf{B}(x, y)$ is an equivalence of categories. We construct an explicit inverse functor by choosing, for each $f \in \mathsf{B}(x, y)$ an isomorphism $\eta_f : \bar{f} \to f$, where $\bar{f}$ is the unique representative in the isomorphism class of f. We impose the convention that if $f = \bar{f}$ happens to be our chosen representative, then $\eta_{\bar{f}} = id_{\bar{f}}$.

One choice of inverse functor $P : \mathsf{B}(x, y) \to \mathsf{B}^s(x, y)$ is defined on objects as the identity, on 1-morphism by $P(f) =: \overline{f}$ and on 2-morphisms as the unique 2-morphism that makes the following diagram commute:

$$\begin{tikzcd}
\bar{f} \arrow[r, dashed, "P(\alpha) =: \overline{\alpha}"] \arrow[d, "\eta_f"'] & \bar{g} \arrow[d, "\eta_g"] \\
f \arrow[r, "\alpha"'] & g
\end{tikzcd}$$

In particular, the compositon $P \circ i = id_{\mathsf{B}^s(x,y)}$ (strict equality) and the η_f provide the components of a natural isomorphism $\eta : i \circ P \to id_{\mathsf{B}(x,y)}$. Strictly speaking, both P and η depend on x and y, but we will omit this dependence from our notation.

[2]In any good inductive definition of n-category one should be able to similarly define the notion of n-skeletal. In this case a bicategory is always k-skeletal for $k \geq 2$, and a skeletal bicategory is k-skeletal for all k.

We equip B^s with identity 1-morphism $P(I_x) \in \mathsf{B}^s(x,x)$ and with compositions:

$$\mathsf{B}^s(y,z) \times \mathsf{B}^s(x,y) \overset{i \times i}{\hookrightarrow} \mathsf{B}(y,z) \times \mathsf{B}(x,y) \overset{c}{\to} \mathsf{B}(x,z) \overset{P}{\to} \mathsf{B}^s(x,z)$$

i.e. the composition of 1-morphisms f and g is given by $P(i(f) \circ i(g))$. The fully-faithful functor i is automatically injective, and so we may identify $f \in \mathsf{B}^s(x,y)$ with its image $i(f) \in \mathsf{B}(x,y)$, and thereby simply write $P(f \circ g)$ for the composition of f and g in B^s. Finally, we must specify the associators and unitors. These are given on components by:

$$a^{\mathsf{B}^s} : P(P(f \circ g) \circ h) \overset{P(\eta * 1)}{\to} P((f \circ g) \circ h) \overset{P(a^{\mathsf{B}})}{\to} P(f \circ (g \circ h)) \overset{P(1 * \eta^{-1})}{\to} P(f \circ P(g \circ h))$$

$$r^{\mathsf{B}^s} : P(f \circ P(I)) \overset{P(1 * \eta)}{\to} P(f \circ I) \overset{P(r^{\mathsf{B}})}{\to} P(f) = f$$

$$\ell^{\mathsf{B}^s} : P(P(I) \circ f) \overset{P(\eta * 1)}{\to} P(I \circ f) \overset{P(\ell^{\mathsf{B}})}{\to} P(f) = f$$

These are easily verified to satisfy the pentagon and triangle identities, and hence give B^s the structure of a bicategory. We will return to these identities after finishing the rest of this proof.

The functors $i : \mathsf{B}^s(x,y) \to \mathsf{B}(x,y)$ and $P : \mathsf{B}(x,y) \to \mathsf{B}^s(x,y)$ assemble into a pair of homomorphisms: $i : \mathsf{B}^s \to \mathsf{B}$ and $P : \mathsf{B} \to \mathsf{B}^s$ which are given as follows: The functor $i = (id, i, \phi^i_{g,f}, \phi^i_x)$ is the identity on the objects and on hom categories is given by the component functors $i : \mathsf{B}^s(x,y) \to \mathsf{B}(x,y)$. The natural transformations ϕ^i are given by $\phi^i_{g,f} = \eta^{-1} : g \circ f \to P(g \circ f)$ and $\phi^i_x = \eta^{-1} : I_x \to P(I_x)$.

Similarly the homomorphism $P = (id, P, \phi^P_{g,f}, \phi^P_x)$ is the identity on objects and on hom categories is given by $P : \mathsf{B}(x,y) \to \mathsf{B}^s(x,y)$. The natural transformations ϕ^P are given by $\phi^P_{g,f} = P(\eta * \eta) : P(P(g) \circ P(f)) \to P(g \circ f)$ and $\phi^i_x = id : P(I_x) \to P(I_x)$. With these choices the composite homomorphism $P \circ i : \mathsf{B}^s \to \mathsf{B}^s$ is precisely the identity homomorphism, as the reader can readily check. In particular the morphisms,

$$P(I_x) \overset{id}{\to} P(I_x) \overset{P(\eta^{-1})}{\to} P(I_x)$$

$$P(g \circ f) = P(P(g) \circ P(f)) \overset{P(\eta * \eta)}{\to} P(g \circ f) \overset{P(\eta^{-1})}{\to} P(g \circ f))$$

are identities.

The reverse composition is the homomorphism $i \circ P = (id, i \circ P, \phi^{i \circ P}_{f,g}, \phi^{i \circ P}_x)$ where these natural transformations are given on components by:

$$\phi^{i \circ P}_{f,g} : \bar{f} \circ \bar{g} \overset{\eta^{-1}}{\to} \overline{\bar{f} \circ \bar{g}} \overset{\overline{\eta * \eta}}{\to} \overline{f \circ g},$$

$$\phi_x^{i \circ P} : I_x \xrightarrow{\eta^{-1}} \overline{I}_x.$$

Recall that $\overline{f \circ g}$ means $P(f \circ g)$. This is not the identity homomorphism, but it is equivalent to the identity by a transformation whose components are $\sigma = (I_x, \eta_f)$. Thus the bicategories B and B^s are equivalent. $\square$

Now let us return to the verification of the pentagon and triangle identities and the choice of associator a^{B^s} and unitors, above. The associator for B^s is a morphism from $\overline{(\overline{f \circ g}) \circ h}$ to $\overline{f \circ (\overline{g \circ h})}$. The definition of a^{B^s} we gave above used a combination of the η 2-morphisms to construct isomorphisms:

$$\overline{(\overline{f \circ g}) \circ h} \cong (f \circ g) \circ h$$
$$\overline{f \circ (\overline{g \circ h})} \cong f \circ (g \circ h)$$

and then used these to pullback the associator of B, $a : (f \circ g) \circ h \to f \circ (g \circ h)$. One might ask the following question: why did we choose these particular isomorphisms? For example the first isomorphism, as specified in the above proof, is the composite:

$$\overline{(\overline{f \circ g}) \circ h} \to \overline{(f \circ g) \circ h} \to (f \circ g) \circ h,$$

but we could have just as easily insisted on using the composite:

$$\overline{(\overline{f \circ g}) \circ h} \to (\overline{f \circ g}) \circ h \to (f \circ g) \circ h.$$

In fact, these isomorphisms are the same, as can be seen from the naturality of η. However more is true. A marked bracketed expression, such as $\overline{(\overline{f \circ g}) \circ h}$ determines a marked binary tree. In this case it is the tree $((-) \boxdot (-)) \boxdot (-)$ and the marked edges are precisely those corresponding to $f \circ g$ and $(f \circ g) \circ h$.

Each marked edge corresponds to a single application of the functor P, which may be removed by composing with a single instance of the η 2-morphisms. Thus a path between marked binary trees corresponds to a particular choice of composite, such as the two considered above. These various compositions are often equal, and we may impose an equivalence relation on the paths between marked binary trees such the two paths are equivalent precisely when the corresponding composites of 2-morphisms are equal. The naturality of the natural transformation η implies that property (ii) of Lemma 3.4.3 is satisfied by this equivalence relation. Similarly, the functoriality of $\circ$ implies both properties

(i) and (iii). Thus, by Lemma 3.4.3 we see that any two paths are equivalent, and hence that any two corresponding composites of the η's coincide.

This implies that given any marked bracketed expression, such as $\overline{(\overline{(\overline{f \circ g}) \circ h}) \circ j}$ used in the verification of the pentagon identity, there is a *canonical* isomorphism with the same bracketed expression, equipped with fewer markings, and moreover these canonical isomorphisms are coherent. With this observation at hand the verification of the pentagon axiom is straightforward.

Lemma 3.4.6. *Every symmetric monoidal bicategory is equivalent to a 1-skeletal symmetric monoidal bicategory.*

Proof. Let B be a symmetric monoidal bicategory, and construct the 1-skeletal bicategory B^s and equivalence of bicategories $\mathsf{B}^s \simeq \mathsf{B}$ as in the previous lemma. We must transfer the symmetric monoidal structure from B to B^s and promote the equivalence $\mathsf{B}^s \simeq \mathsf{B}$ to a symmetric monoidal equivalence.

B^s becomes a symmetric monoidal bicategory when equipped with the following additional structure. The unit $1 \in \mathsf{B}$ is the identical object of B^s (recall B and B^s have the same objects). The tensor product $\otimes^{\mathsf{B}^s}$ on B^s is defined to be the composite homomorphism:

$$\mathsf{B}^s \times \mathsf{B}^s \overset{i \times i}{\hookrightarrow} \mathsf{B} \times \mathsf{B} \overset{\otimes}{\to} \mathsf{B} \overset{P}{\to} \mathsf{B}^s.$$

Thus on objects, $\otimes^{\mathsf{B}^s}$ agrees with $\otimes^{\mathsf{B}}$, and on 1-morphisms and 2-morphisms it is given by $f \otimes^{\mathsf{B}^s} g := \overline{f \otimes g}$. Lemma 3.4.3 again implies the existence of canonical coherent isomorphisms between marked bracketed expressions involving any mixture of $\circ$ and $\otimes$ (property (ii) follows from the naturality of η and properties (i) and (iii) follow from the functoriality of $\circ$ and $\otimes$. Thus we may define α^{B^s} (and all the remaining structure) by conjugating the structure on B by these canonical isomorphisms. The relevant coherence diagrams are then automatically satisfied. Explicitly, we have the following structure transformations:

$$\begin{aligned}
\alpha^{\mathsf{B}^s} &= (\bar{\alpha}_{abc}, \eta^{-1} \circ (1 \otimes \eta^{-1}) \circ \alpha_{fgh} \circ (\eta \otimes 1) \circ \eta) = (\bar{\alpha}_{abc}, \mathrm{can}^{-1} \circ \alpha_{fgh} \circ \mathrm{can}) \\
\ell^{\mathsf{B}^s} &= (\bar{\ell}_a, \ell_f \circ \eta) \\
r^{\mathsf{B}^s} &= (\bar{r}, r_f \circ \eta) \\
\beta^{\mathsf{B}^s} &= (\bar{\beta}, \eta^{-1} \circ \beta_{f,g} \circ \eta)
\end{aligned}$$

together with structure modifications defined by the following pasting diagrams:

$$
\pi^{\mathsf{B}^s} :=
\begin{tikzcd}
 & & (a\otimes b)\otimes(c\otimes d) \arrow[drr, "\alpha"'] \arrow[drr, bend left=60, "\bar{\alpha}"] \arrow[drr, phantom, bend left=30, "\Uparrow"] & & \\
((a\otimes b)\otimes c)\otimes d \arrow[urr, "\alpha"'] \arrow[urr, bend left=60, "\bar{\alpha}"] \arrow[urr, phantom, bend left=30, "\Uparrow"] \arrow[dr, "\alpha\otimes I"] \arrow[dr, bend right=60, "\overline{\overline{\alpha}\otimes\overline{I}}"'] \arrow[dr, phantom, bend right=30, "\Rightarrow"] & & \Uparrow\pi & & a\otimes(b\otimes(c\otimes d)) \\
 & (a\otimes(b\otimes c))\otimes d \arrow[rr, "\alpha"] \arrow[rr, bend right=60, "\bar{\alpha}"'] \arrow[rr, phantom, bend right=30, "\Uparrow"] & & a\otimes((b\otimes c)\otimes d) \arrow[ur, "I\otimes\alpha"] \arrow[ur, bend right=60, "\overline{\overline{I}\otimes\overline{\alpha}}"'] \arrow[ur, phantom, bend right=30, "\Leftarrow"] &
\end{tikzcd}
$$

$$
\mu^{\mathsf{B}^s} :=
\begin{tikzcd}
 & (a\otimes 1)\otimes b \arrow[r, "\alpha"'] \arrow[r, bend left=60, "\overline{\alpha}"] \arrow[r, phantom, bend left=30, "\Downarrow"] & a\otimes(1\otimes b) \arrow[dr, "I\otimes\ell"'] \arrow[dr, bend left=60, "\overline{\overline{I}\otimes\overline{\ell}}"] \arrow[dr, phantom, bend left=30, "\Leftarrow"] & \\
a\otimes b \arrow[ur, "r\otimes I"'] \arrow[ur, bend left=60, "\overline{\overline{r}\otimes\overline{I}}"] \arrow[ur, phantom, bend left=30, "\Rightarrow"] \arrow[rrr, "I_{a\otimes b}"] \arrow[rrr, bend right=40, "\overline{I_{a\otimes b}}"'] \arrow[rrr, phantom, bend right=20, "\Downarrow"] & & \Downarrow\mu & a\otimes b
\end{tikzcd}
$$

$$
\lambda^{\mathsf{B}^s} :=
\begin{tikzcd}
(1\otimes a)\otimes b \arrow[rr, "\ell\otimes I"'] \arrow[rr, bend left=50, "\overline{\overline{\ell}\otimes\overline{I}}"] \arrow[rr, phantom, bend left=25, "\Downarrow"] \arrow[dr, bend right=20, "\alpha"] \arrow[dr, bend right=70, "\overline{\alpha}"'] \arrow[dr, phantom, bend right=45, "\Leftarrow"] & \Downarrow\lambda & a\otimes b \\
 & 1\otimes(a\otimes b) \arrow[ur, bend right=20, "\ell"] \arrow[ur, bend right=70, "\overline{\ell}"'] \arrow[ur, phantom, bend right=45, "\Rightarrow"] &
\end{tikzcd}
\qquad
\rho^{\mathsf{B}^s} :=
\begin{tikzcd}
a\otimes b \arrow[rr, "I\otimes r"'] \arrow[rr, bend left=50, "\overline{\overline{I}\otimes\overline{r}}"] \arrow[rr, phantom, bend left=25, "\Downarrow"] \arrow[dr, bend right=20, "r"] \arrow[dr, bend right=70, "\overline{r}"'] \arrow[dr, phantom, bend right=45, "\Leftarrow"] & \Downarrow\rho & a\otimes(b\otimes 1) \\
 & (a\otimes b)\otimes 1 \arrow[ur, bend right=20, "\alpha"] \arrow[ur, bend right=70, "\overline{\alpha}"'] \arrow[ur, phantom, bend right=45, "\Rightarrow"] &
\end{tikzcd}
$$

$$
R^{\mathsf{B}^s} :=
\begin{tikzcd}[column sep=small]
 & a \otimes (b \otimes c) \arrow[rr, "\beta"'] \arrow[rr, bend left=60, "\overline{\beta}"] \arrow[rr, phantom, bend left=30, "\Downarrow"] & & (b \otimes c) \otimes a \arrow[dr, "\alpha"'] \arrow[dr, bend left=50, phantom, "\Leftarrow"] & \\
(a \otimes b) \otimes c \arrow[ur, "\alpha"'] \arrow[ur, bend left=60, "\overline{\alpha}"] \arrow[ur, phantom, bend left=30, "\Rightarrow"] \arrow[dr, "\beta \otimes I"] \arrow[dr, bend right=60, "\overline{\overline{\beta} \otimes \overline{I}}"'] \arrow[dr, phantom, bend right=30, "\Leftarrow"] & & \Downarrow R & & b \otimes (c \otimes a) \\
 & (b \otimes a) \otimes c \arrow[rr, "\alpha"] \arrow[rr, bend right=60, "\overline{\alpha}"'] \arrow[rr, phantom, bend right=30, "\Downarrow"] & & b \otimes (a \otimes c) \arrow[ur, "I \otimes \beta"] \arrow[ur, bend right=60, "\overline{\overline{I} \otimes \overline{\beta}}"'] \arrow[ur, phantom, bend right=30, "\Rightarrow"] &
\end{tikzcd}
$$

$$
S^{\mathsf{B}^s} :=
\begin{tikzcd}[column sep=small]
 & (a \otimes b) \otimes c \arrow[rr, "\beta"'] \arrow[rr, bend left=60, "\overline{\beta}"] \arrow[rr, phantom, bend left=30, "\Downarrow"] & & c \otimes (a \otimes b) \arrow[dr, "\alpha^*"'] \arrow[dr, bend left=60, "\overline{\alpha}^*"] \arrow[dr, phantom, bend left=30, "\Leftarrow"] & \\
a \otimes (b \otimes c) \arrow[ur, "\alpha^*"'] \arrow[ur, bend left=60, "\overline{\alpha}^*"] \arrow[ur, phantom, bend left=30, "\Rightarrow"] \arrow[dr, "I \otimes \beta"] \arrow[dr, bend right=60, "\overline{\overline{I} \otimes \overline{\beta}}"'] \arrow[dr, phantom, bend right=30, "\Leftarrow"] & & \Downarrow S & & (c \otimes a) \otimes b \\
 & a \otimes (c \otimes b) \arrow[rr, "\alpha^*"] \arrow[rr, bend right=60, "\overline{\alpha}^*"'] \arrow[rr, phantom, bend right=30, "\Downarrow"] & & (a \otimes c) \otimes b \arrow[ur, "\beta \otimes I"] \arrow[ur, bend right=60, "\overline{\overline{\beta} \otimes \overline{I}}"'] \arrow[ur, phantom, bend right=30, "\Rightarrow"] &
\end{tikzcd}
$$

$$\sigma^{\mathsf{B}^s} := \begin{tikzcd}[column sep=large, row sep=large]
a\otimes b \arrow[rr, "I_{a\otimes b}"'] \arrow[rr, bend left=60, "\overline{I}_{a\otimes b}"] \arrow[rr, phantom, bend left=30, "\Downarrow"] \arrow[dr, bend right, "\beta"] \arrow[dr, bend right=90, "\overline{\beta}"'] \arrow[dr, phantom, bend right=60, "\Downarrow"] & \arrow[d, phantom, "\Downarrow\sigma"] & a\otimes b \\
& b\otimes a) \arrow[ur, bend right, "\beta"] \arrow[ur, bend right=90, "\overline{\beta}"'] \arrow[ur, phantom, bend right=60, "\Downarrow"] &
\end{tikzcd}$$

here the unlabeled 2-morphisms are the natural 2-morphisms η. With these structures B^s becomes a symmetric monoidal bicategory.

The inclusion homomorphism $i : \mathsf{B}^s \to \mathsf{B}$ becomes a symmetric monoidal homomorphism when equipped with the following structure. The structure transformations are given by:

$$\chi^i = (I, f\otimes g \overset{\mathrm{can}^{-1}}{\to} \overline{f\otimes g})$$
$$\iota^i = (I, id)$$

and the structure modifications $\omega^i, \gamma^i, \delta^i, u^i$ are given by the canonical 2-morphism such as ω^i below:

$$\begin{tikzcd}
& a\otimes(b\otimes c) \arrow[rr, "I\otimes I"] & & a\otimes(b\otimes c) \arrow[dr, "I"] & \\
(a\otimes b)\otimes c \arrow[ur, "\alpha"] \arrow[dr, "I\otimes I"'] & & \Downarrow \omega^i = \mathrm{can}^{-1} & & a\otimes(b\otimes c) \\
& (a\otimes b)\otimes c \arrow[rr, "I"'] & & (a\otimes b)\otimes c \arrow[ur, "\overline{\alpha}"'] &
\end{tikzcd}$$

Similarly, $P : \mathsf{B} \to \mathsf{B}^s$ becomes a symmetric monoidal homomorphism when

equipped with the structure transformations:

$$\chi^i = (I, \overline{\overline{f} \otimes \overline{g}} \xrightarrow{\text{can}} \overline{f \otimes g})$$
$$\iota^i = (I, id)$$

together with the canonical structure 2-morphisms $\omega^P, \gamma^P, \delta^P, u^P$ such as ω^P given below:

$$\begin{tikzcd}
& a \otimes (b \otimes c) \arrow[r, "I \otimes I"] & a \otimes (b \otimes c) \arrow[dr, "I"] & \\
(a \otimes b) \otimes c \arrow[ur, "\overline{\alpha}"] \arrow[dr, "I \otimes I"'] \arrow[rrr, phantom, "\Downarrow \omega^P = \text{can}"] & & & a \otimes (b \otimes c) \\
& (a \otimes b) \otimes c \arrow[r, "I"'] & (a \otimes b) \otimes c \arrow[ur, "\overline{\alpha}"'] &
\end{tikzcd}$$

The composition of P and i, equipped with these symmetric monoidal structures, can now be readily computed. We have, in both cases, that the resulting compositions are equivalent to the corresponding identity homomorphism, and thus i is an equivalence of symmetric monoidal bicategories. □

Lemma 3.4.7. *Every symmetric monoidal bicategory is equivalent to a skeletal symmetric monoidal bicategory.*

Proof. Let B be a symmetric monoidal bicategory. By the previous lemma, we may assume, without loss of generality, that B is 1-skeletal. Choose for each equivalence class of objects in B a representative object and let B^{sk} denote the full sub-bicategory of B spanned by these objects. B^{sk} is a skeletal bicategory. We will equip B^{sk} with the structure of a symmetric monoidal bicategory and promote the inclusion functor $j : \mathsf{B}^{sk} \to \mathsf{B}$ to a symmetric monoidal equivalence.

Choose for each object $x \in \mathsf{B}$ an equivalence $\xi_x : \overline{x} \to x$ in B, such that $\xi_{\overline{x}} = I_{\overline{x}}$. Since B is 1-skeletal, there exists a unique 1-morphism $\xi^{-1} : x \to \overline{x}$ such that $\xi\xi^{-1}$ and $\xi^{-1}\xi$ are identity 1-morphisms (strict equality). $j : \mathsf{B}^{sk} \to \mathsf{B}$ is an equivalence of bicategories and an inverse equivalence is given by $Q : \mathsf{B} \to \mathsf{B}^{sk}$ which is defined as follows:

– For objects $x \in \mathsf{B}$, $Q(x) = \overline{x}$.

– For 1-morphisms $f : x \to y$, we have $Q(f) = \overline{f} := \overline{x} \xrightarrow{\xi} x \xrightarrow{f} y \xrightarrow{\xi^{-1}} \overline{y}$. Note that because B is 1-skeletal the order of this composition is irrelevant.

– For 2-morphisms $\alpha : f \to g$, we define $Q(\alpha) = \overline{\alpha}$ to be the horizontal composition $[id_\xi * \alpha] * id_{\xi^{-1}}$.

Just as before, we have that $Q \circ j$ is the identity homomorphism on B^{sk} and that $j \circ Q$ is equivalent to the identity homomorphism on B, with ξ providing the structure of that equivalence.

B^{sk} becomes a symmetric monoidal bicategory when equipped with the following additional structure. The unit is given by $\overline{1} \in \mathsf{B}^{sk}$. The tensor product $\otimes^{\mathsf{B}^{sk}}$ on B^{sk} is defined to be the composite homomorphism:

$$\mathsf{B}^{sk} \times \mathsf{B}^{sk} \xhookrightarrow{j \times j} \mathsf{B} \times \mathsf{B} \xrightarrow{\otimes} \mathsf{B} \xrightarrow{Q} \mathsf{B}^{sk}.$$

Thus on objects, $\otimes^{\mathsf{B}^{sk}}$ is given by $x \otimes^{\mathsf{B}^{sk}} y = \overline{x \otimes y}$, and similarly on 1-morphisms it is given by $\overline{f \otimes g}$ and on 2-morphisms by $\overline{\zeta \otimes \kappa}$. Lemma 3.4.3 implies the existence of canonical coherent 1-morphisms between marked bracketed expressions involving the objects of B^{sk} and $\otimes$. Thus we may define the symmetric monoidal structure on B^{sk} by conjugating the corresponding structures on B, by these canonical 1-morphisms. For example the pentagonator $\pi^{\mathsf{B}^{sk}}$ is given by the following pasting diagram:

$$\begin{tikzcd}
 & & & \overline{(\overline{a\otimes b})\otimes(\overline{c\otimes d})} \arrow[d, "\text{can.}"'] \arrow[drrr, "\overline{\alpha}"] & & & \\
\overline{(\overline{(\overline{a\otimes b})\otimes c})\otimes d} \arrow[urrr, "\overline{\alpha}"] \arrow[dr, "\text{can.}"] \arrow[dddr, "\overline{\alpha}\otimes I"'] & & & (a\otimes b)\otimes(c\otimes d) \arrow[drr, "\alpha"] & & & \overline{a\otimes(\overline{b\otimes(\overline{c\otimes d})})} \\
 & ((a\otimes b)\otimes c)\otimes d \arrow[urr, "\alpha"] \arrow[dr, "\alpha\otimes I"] & & & & a\otimes(b\otimes(c\otimes d)) \arrow[ur, "\text{can.}"] & \\
 & & (a\otimes(b\otimes c))\otimes d \arrow[rr, bend right, "\alpha"'] & & a\otimes((b\otimes c)\otimes d) \arrow[ur, "I\otimes\alpha"] & & \\
 & \overline{(a\otimes(\overline{b\otimes c}))\otimes d} \arrow[ur, "\text{can.}"] \arrow[rrrr, bend right, "\overline{\alpha}"'] & & & & \overline{a\otimes((\overline{b\otimes c})\otimes d)} \arrow[ul, "\text{can.}"'] \arrow[uuur, "I\otimes\overline{\alpha}"'] &
\arrow[from=2-1, to=2-4, phantom, "\#"]
\arrow[from=2-4, to=2-7, phantom, "\#"]
\arrow[from=3-2, to=3-6, phantom, "\Uparrow\pi"]
\arrow[from=2-1, to=4-3, phantom, "\#"]
\arrow[from=2-7, to=4-5, phantom, "\#"]
\arrow[from=4-4, to=5-4, phantom, "\#"]
\end{tikzcd}$$

here the symbol "#" means the identity 2-morphism, i.e. that the diagram commutes one the nose. It is straightforward to check that the relevant coherence diagrams are automatically satisfied.

We can promote j to a symmetric monoidal homomorphism as well. We equip it with the following structure transformations and modifications: χ^j is the transformation which in components is given by the canonical 1-morphism:

$$\chi^j_{a,b} : a \otimes b \to \overline{a \otimes b}.$$

There is also a component 2-morphism for χ. This 2-morphism can be taken to be the identity, which fills the following pasting diagram (when appropriately bracketed)[3]:

$$\begin{tikzcd}
\overline{a \otimes b} \arrow[d, "(\chi^j_{a,b})^{-1}"'] & a \otimes b \arrow[l, "\chi^j_{a,b}"'] \arrow[ddd, "f \otimes g"] \\
a \otimes b \arrow[d, "f \otimes g"'] & \\
a' \otimes b' \arrow[d, "\chi^j_{a',b'}"'] & \\
\overline{a' \otimes b'} & a' \otimes b' \arrow[l, "\chi^j_{a',b'}"]
\end{tikzcd}$$

Similarly $\iota^j : 1 \to \overline{1}$ is the canonical 1-morphism and we can take $\omega^j, \gamma^j, \delta^j$ and u^j to be identity 2-morphisms.

Similarly Q can be made into a symmetric monoidal homomorphism by equipping it with the transformations $\chi^Q_{a,b} = \text{can.} : \overline{\overline{a} \otimes \overline{b}} \to \overline{a \otimes b}$ and $\iota^Q = I_{\overline{1}}$. All the additional structure 2-morphisms can be taken to be identities. Q and j are inverse equivalences of symmetric monoidal bicategories. $\square$

Lemma 3.4.8. *Let $F : \mathsf{B} \to \mathsf{C}$ be an equivalence between skeletal bicategories. Then there exists a canonical inverse homomorphism $F^{-1} : \mathsf{C} \to \mathsf{B}$ such that $F^{-1} \circ F = id_{\mathsf{B}}$ and $F \circ F^{-1} = id_{\mathsf{C}}$ on the nose.*

[3] Actually the bracketing which allows this to be filled by the identity 2-morphism is not compatible with the bracketing needed to define $\chi^j_{f,g}$, however there is a canonical $\chi^j_{f,g}$ given by rebracketing the identity 2-morphism appropriately.

Proof. Since B and C are skeletal and since F is an equivalence, the components F_0, F_1, F_2 are all bijections with inverses $F_i^{-1} : \mathsf{C}_i \to \mathsf{B}_i$. Let $\eta_{f,g} = F^{-1}((\phi^F{}_{F^{-1}(f),F^{-1}(g)})^{-1})$ for all 1-morphisms $f, g \in \mathsf{C}$, and let $\eta_a = F^{-1}((\phi^F_{F^{-1}(a)})^{-1}) : I_{F^{-1}(a)} \to F^{-1}(I_a)$ for all objects $a \in \mathsf{C}$. Applying the maps F_i^{-1} to the coherence diagrams of (F, ϕ) yield precisely the necessary coherence diagrams for (F^{-1}, η) to be a homomorphism. The compositions of F and F^{-1} can easily be checked to be the identity homomorphisms. □

Up to this point we have mainly been describing symmetric monoidal bicategories with the language of pasting diagrams. The proof of the next lemma, however, will need to make use of *mates*, as described in Appendix B.4. The language of string diagrams is far better suited to this task, and so we will need to use it below. The unfamiliar reader should consult the Appendix.

Lemma 3.4.9. *A symmetric monoidal homomorphism $F : \mathsf{M} \to \mathsf{M}'$ between skeletal symmetric monoidal bicategories is an equivalence if and only if it is an equivalence of underlying bicategories, i.e. the following properties are satisfied:*

1. *(essentially surjective on objects) It is surjective on π_0.*

2. *(essentially full on 1-morphism) $F : \pi_0\mathsf{M}(x, y) \to \pi_0\mathsf{M}'(Fx, Fy)$ is surjective.*

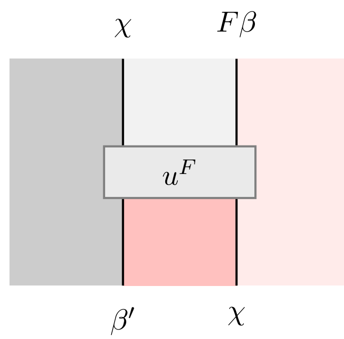
3. *(fully-faithful on 2-morphisms) $F : \mathsf{M}(x, y) \to \mathsf{M}'(Fx, Fy)$ is fully-faithful.*

Proof. The necessity of F to be an equivalence of underlying bicategories is clear. We construct the inverse symmetric monoidal equivalence as follows: First construct the canonical inverse equivalence (F^{-1}, η) of bicategories from Lemma 3.4.8. Then we construct the remaining coherence data for F^{-1} by applying F^{-1} to certain mates of the coherence data of F. Specifically, $\chi^{F^{-1}}$ is given by whiskering $(\chi^F)^{-1}$ by F^{-1} and $F^{-1} \times F^{-1}$. Note that "$(\chi^F)^{-1}$" makes sense since we are dealing with skeletal bicategories. On objects we have $\chi^{F^{-1}}_{x,y} := F^{-1}((\chi^F_{F^{-1}x,F^{-1}y})^{-1})$. We can similarly, define $\iota^{F^{-1}}$ as $F^{-1}((\iota^F)^{-1})$.

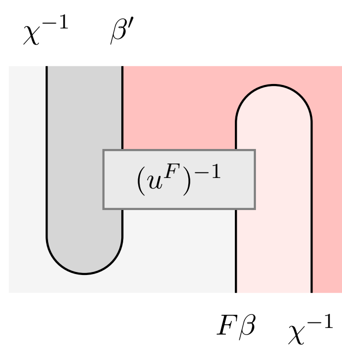
The higher coherence data is constructed as follows. Applying u^F to the objects $F^{-1}(x)$ and $F^{-1}(y)$ yields the following 2-morphism:

$$\begin{tikzcd}
& F(F^{-1}(y) \otimes F^{-1}(x)) \arrow[dr, "F\beta"] & \\
y \otimes x \arrow[ur, "\chi"] \arrow[dr, "\beta'"'] \arrow[rr, phantom, "\Downarrow u^F"] & & F(F^{-1}(x) \otimes F^{-1}(y)) \\
& x \otimes y \arrow[ur, "\chi"'] &
\end{tikzcd}$$

in string diagrams we can represent this 2-morphism as:



We can then form the following mate:

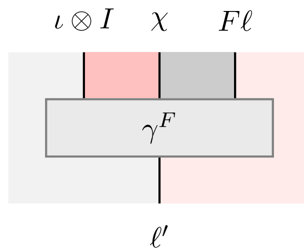
$u^* =$ 

applying F^{-1} to u^* yields $u^{F^{-1}}_{x,y}$.

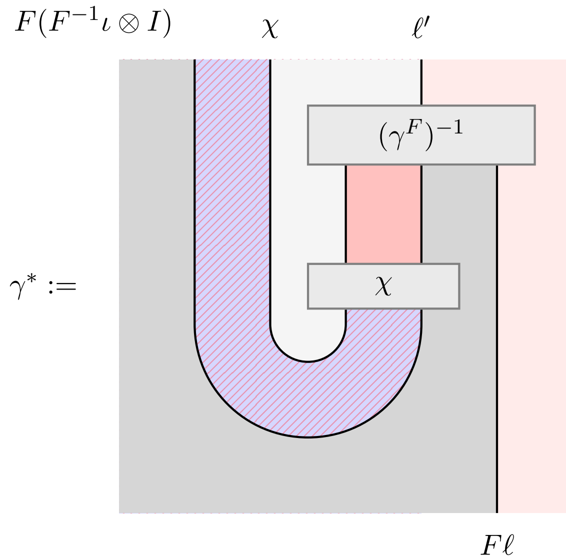
Similarly we may construct $\gamma^{F^{-1}}, \delta^{F^{-1}}$, and $\omega^{F^{-1}}$. Applying γ^F to the object $F^{-1}x$ yields the 2-morphism:

$$\begin{tikzcd}
F(1) \otimes x = F(1) \otimes F(F^{-1}x) \arrow[r, "\chi"] & F(1 \otimes F^{-1}x) \arrow[d, "F\ell"] \\
1' \otimes x \arrow[u, "\iota \otimes I"] \arrow[r, bend right, "\ell'"'] \arrow[ur, phantom, "\Downarrow \gamma^F"] & x = F(F^{-1}x)
\end{tikzcd}$$

which we can represent by the string diagram:



Now we can form the following 2-morphism:



where the 2-morphism labeled "χ" is part of the structure making χ a transformation. It fills the following diagram:

$$\begin{tikzcd}
F(F^{-1}1') \otimes F(F^{-1}x) \arrow[r, "\iota \otimes I"] \arrow[d, "\chi"'] \arrow[dr, phantom, "\Downarrow"] & F(1) \otimes F(F^{-1}x) \arrow[d, "\chi"] \\
F(F^{-1}(1') \otimes F^{-1}(x)) \arrow[r, "F(F^{-1}(\iota) \otimes F^{-1}(I_x))"'] & F(1 \otimes F^{-1}x)
\end{tikzcd}$$

where we have identified $\iota \otimes I = F(F^{-1}(\iota)) \otimes F(F^{-1}(I_x))$. $\gamma^{F^{-1}}$ is obtained by applying F^{-1} to γ^*. The construction of $\delta^{F^{-1}}$ and $\omega^{F^{-1}}$ are completely analogous.

Using string diagrams it is relatively easy, but quite tedious, to verify that these do indeed satisfy the necessary axioms to make F^{-1} into a symmetric monoidal homomorphism.

Moreover a direct calculation shows that the compositions $F \circ F^{-1}$ and $F^{-1} \circ F$ are the respective identity symmetric monoidal homomorphisms (equality on the nose). □

Theorem 3.4.10 (Whitehead's Theorem for Symmetric Monoidal Bicategories)**.** *Let* B *and* C *be symmetric monoidal bicategories. A symmetric monoidal homomorphism* $F : \mathsf{B} \to \mathsf{C}$ *is a symmetric monoidal equivalence if and only if it is an equivalence of underlying bicategories.*

Proof. The necessity of F to be an equivalence of underlying bicateogries is clear. Conversely, let F be a symmetric monoidal homomorphism, which is an equivalence of underlying bicategories. By Lemma 3.4.7, B is equivalent as a symmetric monoidal bicategory to a skeletal symmetric monoidal bicategory B^{sk}, and similarly C is equivalent to a skeletal symmetric monoidal bicategory C^{sk}. The symmetric monoidal homomorphism F is a symmetric monoidal equivalence if and only if the induced homomorphism $F^{sk} : \mathsf{B}^{sk} \to \mathsf{C}^{sk}$ is a symmetric monoidal equivalence. Our assumption that F is an equivalence of underlying bicategories, together with Lemma 3.4.9, ensures this is the case.

□

3.5 Freely Generated Symmetric Monoidal Bicategories

In this section we introduce freely generated symmetric monoidal bicategories. Roughly, the idea is that given certain *generating data*, G, we will be able to produce a symmetric monoidal bicategory, $\mathsf{F}(G)$, which has the property that the bicategory of symmetric monoidal homomorphism out of $\mathsf{F}(G)$ into M is equivalent to a bicategory of "G-shaped" data in the target symmetric monoidal biategory M. The generating data will consist of, among other things, three sets G_0, G_1 and G_2 called *generating objects*, *generating 1-morphism*, and *generating 2-morphisms*, respectively, and $\mathsf{F}(G)$ should be regarded as the free symmetric monoidal bicategory whose objects are generated by G_0, whose 1-morphisms are generated by G_1 and whose 2-morphisms are generated by G_2.

If we only cared about the case when G is a bicategory, then the bicategory $\mathsf{F}(G)$ would be relatively straightforward to construct.[1] However, we wish to allow more general generating data which may not be easily organized into a bicategory. For example, we

[1]For example we could consider F as the (weak) left adjoint to the forgetful (tri-)homomorphism $\mathsf{SymBicat} \to \mathsf{Bicat}$.

will wish to allow our generating 1-morphisms to have sources and targets which are *not* part of the generating objects, but merely consequences, (bracketed) words in the generating objects like "$(w \otimes x) \otimes (y \otimes z)$". Thus to give the precise definition of generating morphisms, we must already have some rudimentary knowledge of what the objects of $\mathsf{F}(G)$ will be. A similar problem persists for generating 2-morphisms. Consequently, we are lead to the following sequence of recursive definitions.

Definition 3.5.1. A *0-truncated generating datum* consists of a set G_0. Given such a set, a *binary word* in G_0 is a binary tree with leaves labeled by elements of G_0, i.e. we have the following recursive definition: the word (1) and the words (a) for $a \in G_0$ are binary words in G_0. If u and v are binary words in G_0 then so is $u \otimes v = (u) \otimes (v)$. $\diamond$

Definition 3.5.2. A *1-truncated generating datum* consists of a set G_0, a set G_1, and a pair of maps $s, t : G_1 \to BW(G_0)$, where $BW(G_0)$ denotes the set of binary words in G_0. Given a 1-truncated datum G_1, then a *binary word* in G_1, which we will define momentarily, has a *source* and a *target*, which are both binary words in G_0. *Binary words* in G_1 are defined recursively as follows:

- If $f \in G_1$, then (f) is a binary word in G_1, with source $s(f)$ and target $t(f)$.
- If u, v, w are binary words in G_0 and x is a symbol from Table 3.4, then (x) is a binary word in G_1 with source and target as listed in Table 3.4.
- If u, v, w are binary words in G_0 and x is a symbol from Table 3.4, then (x^*) is a binary word in G_1 with source and target the opposite of those listed in Table 3.4.
- If f and g are two binary words in G_1, then $f \otimes g$ is a binary word in G_1 with source $s(f) \otimes s(g)$ and target $t(f) \otimes t(g)$.

We also define *binary sentences* in G_1, which again have sources and targets which are binary words in G_0. These are defined recursively as follows:

- If f is a binary word in G_1, then (f) is a binary sentence in G_1 with source $s(f)$ and target $t(f)$.
- If f and g are binary sentences in G_1 such that $s(f) = t(g)$, then $(f) \circ (g)$ is a binary sentence in G_1 with source $s(g)$ and target $t(f)$.

Table 3.4: Binary Words in 1-Truncated Generating Data.

Symbol	Source	Target
I_u	u	u
$\alpha_{u,v,w}$	$(u \otimes v) \otimes w$	$u \otimes (v \otimes w)$
ℓ_u	$1 \otimes u$	u
r_u	u	$u \otimes 1$
$\beta_{u,v}$	$u \otimes v$	$v \otimes u$

– If f and g are binary sentences in G_1, then $f \otimes g$ is a binary sentence in G_1 with source $s(f) \otimes s(g)$ and target $t(f) \otimes t(g)$.

◇

Definition 3.5.3. A *generating datum* G, consists of a 1-truncated generating datum (G_0, G_1, s, t) and a set G_2 together with a pair of maps $s, t : G_2 \to BS(G_0, G_1, s, t)$ where $BS(G_0, G_1, s, t)$ denotes the set of binary sentences in G_1. These maps are required to satisfy:

$$s(s(\epsilon)) = s(t(\epsilon)),$$
$$t(s(\epsilon)) = t(t(\epsilon)),$$

for all $\epsilon \in G_2$. Given a generating datum G_1, then a *binary word* in G_2 is defined recursively as follows. A binary word has a source and a target which are binary sentences in G_1.

– If $\epsilon \in G_2$ then (ϵ) is a binary word in G_2 with source $s(\epsilon)$ and target $t(\epsilon)$.

– If a, a', b, c, d are binary words in G_0, $f : a \to b, g, h,$ f' and g' are binary sentences in G_1 such that $s(f) = t(f')$, $s(f') = t(f'')$ and $s(g) = t(g')$ and x is a symbol from Table 3.5, then (x) is a binary word in G_2 with source and target as listed in Table 3.5.

– If a, a', b, c, d are binary words in G_0, $f : a \to b, g, h,$ f' and g' are binary sentences in G_1 such that $s(f) = t(f')$, $s(f') = t(f'')$ and $s(g) = t(g')$ and x is a symbol from Table 3.5, then (x^{-1}) is a binary word in G_2 with source and target the opposite of those listed in Table 3.5.

– If u and v are binary words in G_2, then $u \otimes v$ is a binary word in G_2 with source $s(u) \otimes s(v)$ and target $t(u) \otimes t(v)$.

Table 3.5: Binary Words in 2-Truncated Generating Data.

Symbol	Source	Target
id_f	f	f
$a^c_{f,f',f''}$	$(f \circ f') \circ f''$	$f \circ (f' \circ f'')$
r^c_f	$f \circ I_a$	f
ℓ^c_f	$I_b \circ f$	f
η_f	I_a	$f^* \circ f$
ε_f	$f \circ f^*$	I_b
$\phi^{\otimes}_{(f,g),(f',g')}$	$(f \otimes g) \circ (f' \otimes g')$	$(f \circ f') \otimes (g \circ g')$
$\phi^{\otimes}_{a,a'}$	$I_{a \otimes a'}$	$I_a \otimes I_{a'}$
$\alpha_{f,g,h}$	$(f \otimes g) \otimes h$	$f \otimes (g \otimes h)$
ℓ_f	$I_1 \otimes f$	f
r_f	f	$f \otimes I_1$
$\beta_{f,g}$	$f \otimes g$	$g \otimes f$
$\pi_{a,b,c,d}$	$[(I \otimes \alpha) \circ \alpha] \circ (\alpha \otimes I)$	$\alpha \circ \alpha$
$\mu_{a,b}$	$[I_a \otimes \ell_b) \circ \alpha_{a,1,b}] \circ (r_a \otimes I_b)$	$I_{a \otimes b}$
$\lambda_{a,b}$	$\ell_a \otimes I_b$	$\ell_{a \otimes b} \circ \alpha_{1,a,b}$
$\rho_{a,b}$	$I_a \otimes r_b$	$\alpha_{a,b,1} \circ r_{a \otimes b}$
$R_{a,b,c}$	$[\alpha_{b,c,a} \circ \beta_{a,b \otimes c}] \circ \alpha_{a,b,c}$	$[(I_b \otimes \beta_{a,c}) \circ \alpha_{b,a,c}] \circ (\beta_{a,b} \otimes I_c)$
$S_{a,b,c}$	$[\alpha^*_{c,a,b} \circ \beta_{a \otimes b,c}] \circ \alpha^*_{a,b,c}$	$[(\beta_{a,c} \otimes I_b) \circ \alpha^*_{a,c,b}] \circ (I_a \otimes \beta_{b,c})$
$\sigma_{a,b}$	$I_{a \otimes b}$	$\beta_{b,a} \circ \beta_{a,b}$

Binary sentences in G_2 are defined recursively as follows:

– If ϵ is a binary word in G_2 then (ϵ) is a binary sentence in G_2 with source $s(\epsilon)$ and target $t(\epsilon)$.

– If u and v are binary sentences in G_2 such that $t(t(v)) = s(s(u))$, then $u * v$ is a binary word in G_2 with source $s(u) \circ s(v)$ and target $t(u) \circ t(v)$.

– If u and v are binary sentences in G_2, then $u \otimes v$ is a binary sentence in G_2 with source $s(u) \otimes s(v)$ and target $t(u) \otimes t(v)$.

Paragraphs in G_2 are defined recursively as follows:

– If ϵ is a binary sentence in G_2 then (ϵ) is a paragraph in G_2 with source $s(\epsilon)$ and target $t(\epsilon)$.

– If p and p' are paragraphs in G_2, such that $t(t(p')) = s(s(p))$, then $(p) * (p')$ is a paragraph in G_2 with source $s(p) \circ s(p')$ and target $t(p) \circ t(p')$.

– If p and p' are paragraphs in G_2, then $(p) \otimes (p')$ is a paragraph in G_2 with source $s(p) \otimes s(p')$ and target $t(p) \otimes t(p')$.

– If $p_0, p_1, \cdots p_k$ is a composable sequence of paragraphs in G_2, i.e. $s(p_{i-1}) = t(p_i$ for $1 \leq i \leq k$, then the (non-binary) word $p_0 p_1 \cdots p_k$ is a paragraph. We also write this paragraph as $p_0 \circ p_1 \circ \cdots \circ p_k$.

$\diamond$

Definition 3.5.4. Let $G = (G_2 \rightrightarrows G_1 \rightrightarrows G_0)$ be a generating datum. Then define $\mathsf{F}(G)$ as the following symmetric monoidal bicategory. The objects of $\mathsf{F}(G)$ consist of the binary words in G_0. The 1-morphisms consist of the binary sentences in G_1 and the 2-morphisms consist of equivalence classes of paragraphs in G_2. The equivalence relation on paragraphs is the finest such that:

– If p is a paragraph with source f and target g, and p^{-1} is also a paragraph, then $pp^{-1} \sim id_g$ and $p^{-1}p \sim id_f$.

– $a^c, r^c, \ell^c, \phi^{\otimes}_{(f,g),(f',g')}, \phi^{\otimes}_{a,a'}, \alpha_{f,g,h}, \ell_f, r_f$, and $\beta_{f,g}$ are natural. This means that for each of these morphisms a certain square commutes. For example, consider binary sentences $f, \tilde{f} : a \to b$ and any paragraph $\xi : f \to \tilde{f}$. If r^c is natural then the following square commutes:

$$\begin{tikzcd}
f \circ I_a \arrow[r, "r^c_f"] \arrow[d, "\xi * id_{I_a}"'] & f \arrow[d, "\xi"] \\
\tilde{f} \circ I_a \arrow[r, "r^c_{\tilde{f}}"'] & \tilde{f}
\end{tikzcd}$$

which is to say that the paragraphs $\xi \circ r^c_f$ and $r^c_{\tilde{f}} \circ (\xi * id_{I_a})$ are equivalent. Naturality for $a^c, \ell^c, \phi^{\otimes}_{(f,g),(f',g')}, \phi^{\otimes}_{a,a'}, \alpha_{f,g,h}, \ell_f, r_f$, and $\beta_{f,g}$ in similar.

– a^c, r^c, ℓ^c satisfy the pentagon and triangle axioms.

– $\otimes, \alpha, \ell, r, \beta, \pi, \mu, \lambda, \rho, R, S, \sigma$ satisfy the axioms for symmetric monoid bicategories.

  That is the equations (TA1), (TA2) and (TA3) of [GPS95] are satisfied, the equations (BA1), (BA2), (BA3), (BA4), (SA1) and (SA2) of [McC00] are satisfied and equation (SMA) is satisfied, see Definition 3.2.1.

– The equivalence relation is closed under $\otimes, *$ and concatenation $\circ$ (i.e. composition).

Vertical composition of 2-morphisms is given by concatenation. The composition of 1-morphisms is given by $\circ$ and the horizontal composition of 2-morphisms is given by $*$. $\diamond$

Let M be a symmetric monoidal bicategory. Given a generating datum, we will introduce an auxiliary bicategory $G(\mathsf{M})$ of "G-data" in M. Roughly, the objects of $G(\mathsf{M})$ consist of an assignment of an object of M for each element of G_0, a 1-morphisms in M for each element of G_1, and a 2-morphism in M for each element of G_2. These are required to be compatible with the source and target maps. For example suppose that $G_0 = \{x, y, z\}$, $G_1 = \{f\}$, and that the source of f is $x \otimes (y \otimes z)$. If we are given an assignment of objects of M for each element G_0 (i.e. a map $G_0 \to \mathsf{M}_0$), then we get a canonical extension to all binary words in G_0 (i.e. an extension to $BW(G_0) \to \mathsf{M}_0$). This extension is given be taking the binary word in G_0 and evaluating it in the symmetric monoidal becategory M. The 1-morphism that we assign to f must now have a source which is exactly the value of $x \otimes (y \otimes z)$, evaluated in M. A similar statement holds for the targets of generating 1-morphisms and for 2-morphisms as well.

Definition 3.5.5. Let $G = (G_2 \rightrightarrows G_1 \rightrightarrows G_0)$ be a generating datum and let M be a symmetric monoidal bicategory. Define the bicategory $G(\mathsf{M})$ as follows:

– The objects of $G(\mathsf{M})$ consist of a triple of maps:

$$\phi : (G_0, G_1, G_2) \to (\mathsf{M}_0, \mathsf{M}_1, \mathsf{M}_2).$$

  Given such a triple, we have a canonical extension to a triple of maps:

$$\tilde{\phi} : (BW(G_0), BS(G_1), P(G_2)) \to (\mathsf{M}_0, \mathsf{M}_1, \mathsf{M}_2),$$

  where $BW(G_0)$ denotes the set of binary words in G_0, $BS(G_1)$ denotes the set of binary sentences in G_1, and $P(G_2)$ denotes the paragraphs in G_2. This extension

map is given by using ϕ to obtain a corresponding expression in M and evaluating it in M using the symmetric monoidal structure on M. The triple ϕ is required to satisfy the conditions:

1. $\tilde{\phi}(s(f)) = s(\phi(f))$ and $\tilde{\phi}(t(f)) = t(\phi(f))$ for all $f \in G_1$,
2. $\tilde{\phi}(s(\epsilon)) = s(\phi(\epsilon))$ and $\tilde{\phi}(t(\epsilon)) = t(\phi(\epsilon))$ for all $\epsilon \in G_2$.

– The 1-morphisms of $G(\mathsf{M})$ (from ϕ_0 to ϕ_1, say) consist of pairs of maps,

$$\psi : (G_0, G_1) \to (\mathsf{M}_1, \mathsf{M}_2),$$

These maps are required to satisfy a number of conditions. Given such a pair, we have a canonical extension $\tilde{\psi} : BW(G_0) \to \mathsf{M}_1$ given by evaluation in M. We require for all $a \in G_0$ and for all $f \in G_1$ that,

$$\begin{aligned} s(\psi(a)) &= \phi_0(a) \\ t(\psi(a)) &= \phi_1(a) \\ s(\psi(f)) &= \phi_1(f) \circ \tilde{\psi}(s(f)) \\ t(\psi(f)) &= \tilde{\psi}(t(f)) \circ \phi_0(f). \end{aligned}$$

We can also extend ψ to a map $\tilde{\psi} : BS(G_1) \to \mathsf{M}_2$, also satisfying this property. This extension is defined inductively as follows:

– $\tilde{\psi} = \psi$ on the elements of $G_1 \subset BS(G_1)$,

– $\tilde{\psi}(I_u)$ is the canonical 1-morphism:

$$(r^c)^{-1} \circ \ell : I_{\tilde{\phi}_1(u)} \circ \psi(u) \to \psi(u) \to \psi(u) \circ I_{\tilde{\phi}_0(u)}.$$

– For ℓ, r, α and β, $\tilde{\psi}$ is defined to be the structure 2-morphisms which realize ℓ, r, α and β as transformations. For example, $\tilde{\psi}(\alpha)$ is defined to be $\alpha^{-1}_{\tilde{\psi}(u),\tilde{\psi}(v),\tilde{\psi}(w)}$, which fills the following diagram:

$$\begin{tikzcd} (\phi_0 u \otimes \phi_0 v) \otimes \phi_0 w \arrow[r, "\alpha"] \arrow[d, "\psi_{(u\otimes v)\otimes w} = (\psi(u) \otimes \psi(v)) \otimes \psi(w)"'] \arrow[dr, phantom, "\Rightarrow"] & \phi_0 u \otimes (\phi_0 v \otimes \phi_0 w) \arrow[d, "\psi(u) \otimes (\psi(v) \otimes \psi(w))"] \\ (\phi_1 u \otimes \phi_1 v) \otimes \phi_1 w \arrow[r, "\alpha"'] & \phi_0 u \otimes (\phi_0 v \otimes \phi_0 w) \end{tikzcd}$$

The values of $\tilde{\psi}(\ell), \tilde{\psi}(r)$, and $\tilde{\psi}(\beta)$ are defined similarly, and this determines $\tilde{\psi}$ on binary words in G_1 of length one.

– We inductively define $\tilde{\psi}$ on binary words of longer length, by defining $\tilde{\psi}(f \otimes f')$ as the composition:

$$\begin{aligned}[]
[\phi_1 f \otimes \phi_1 f'] \circ [\psi a \otimes \psi a'] &\to [\phi_1 f \circ \psi a] \otimes [\phi_1 f' \circ \psi a'] \\
&\overset{\tilde{\psi} f \otimes \tilde{\psi} f'}{\to} [\psi b \circ \phi_0 f] \otimes [\psi b' \circ \phi_1 f'] \\
&\to [\psi b \otimes \psi b'] \circ [\phi_0 f \otimes \phi_1 f']
\end{aligned}$$

– Finally, to define $\tilde{\psi}$ of all binary sentences in G_1, we must define it on compositions. If f and g are binary sentences in G_1 such that $s(f) = t(g)$, then $\tilde{\psi}(f \circ g)$ is given by the diagram

$$\begin{tikzcd}
(\phi_1 g \circ \phi_1 f) \circ \tilde{\psi}_a \arrow[rr, dashed, "\tilde{\psi}_{g \circ f}"] \arrow[d, "a^{\mathsf{M}}"'] & & \tilde{\psi}_c \circ (\phi_0 g \circ \phi_0 f) \\
\phi_1 g \circ (\phi_1 f \circ \tilde{\psi}_a) \arrow[d, "id_{\phi_1 g} * \tilde{\psi}_f"'] & & (\tilde{\psi}_c \circ \phi_0 g) \circ \phi_0 f \arrow[u, "a^{\mathsf{M}}"'] \\
\phi_1 g \circ (\tilde{\psi}_b \circ \phi_0 f) \arrow[rr, "(a^{\mathsf{M}})^{-1}"'] & & (\phi_0 g \circ \tilde{\psi}_b) \circ \phi_0 f \arrow[u, "\tilde{\psi}_g * id_{\phi_o f}"']
\end{tikzcd}$$

Thus we have formed an extension $\tilde{\psi} : BS(G_2) \to \mathsf{M}_2$. For ψ to be a 1-morphism of $G(\mathsf{M})$ we require a further axiom. For all $\epsilon \in G_2$ (with source binary sentence f and target g) we require that the naturality square commutes:

$$\begin{tikzcd}
\phi_1(f) \circ \tilde{\psi}(a) \arrow[r, "\tilde{\psi}(f)"] \arrow[d, "\phi_1(\epsilon) * id_{\tilde{\psi}(a)}"'] & \tilde{\psi}(b) \circ \phi_0(f) \arrow[d, "id_{\tilde{\psi}(b)} * \phi_0(\epsilon)"] \\
\phi_1(g) \circ \tilde{\psi}(a) \arrow[r, "\tilde{\psi}(g)"'] & \tilde{\psi}(b) \circ \phi_0(g)
\end{tikzcd}$$

– The 2-morphism of $G(\mathsf{M})$ (sat from $\psi : \phi_0 \to \phi_1$ to $\theta : \phi_0 \to \phi_1$) are given by certain maps $m : G_0 \to \mathsf{M}_2$. These maps are required to satisfy several properties. First, we require that $s(m(a)) = \psi(a)$ and that $t(m(a)) = \theta(a)$. Such a map as a canonical

extension $m : BW(G_0) \to \mathsf{M}_2$ (again given by evaluation in M), and we further require that for all $f \in G_1$, the following diagram commutes:

$$\begin{tikzcd} \phi_1(f) \circ \psi(a) \arrow[r, "id * m(a)"] \arrow[d, "\psi(f)"'] & \phi_1(f) \circ \theta(a) \arrow[d, "\theta(f)"] \\ \psi(b) \circ \phi_0(f) \arrow[r, "m(b) * id"'] & \theta(b) \circ \phi_0(f) \end{tikzcd}$$

The vertical and horizontal compositions of 2-morphisms in $G(\mathsf{M})$ are given point wise in M, and the composition of 2-morphisms in $G(\mathsf{M})$ is given by the composition of transformations as in Definition B.1.9. The associators and unitors of M induce associators and unitors for $G(\mathsf{M})$. $\diamond$

The canonical extensions considered in the above definitions give the data of the homomorphisms, transformations, and modifications of $\mathsf{SymBicat}(\mathsf{F}(G), \mathsf{M})$. Indeed $\tilde{\phi}$, together with the trivial coherence structure $(\chi, \iota, \omega, \gamma, \delta, u)$ gives a strict symmetric monoidal homomorphism, $\tilde{\psi}$ (with trivial Π and M) gives a strict symmetric monoidal transformation between the $\tilde{\phi}_i$, and m gives a similar symmetric monoidal modification. In constructing $G(\mathsf{M})$ we have simultaneously built a canonical homomorphism:

$$j : G(\mathsf{M}) \to \mathsf{SymBicat}(\mathsf{F}(G), \mathsf{M}).$$

In fact, the composition in $G(\mathsf{M})$ is given by precisely the composition in $\mathsf{SymBicat}(\mathsf{F}(G), \mathsf{M})$, so that this homomorphism is strict.

Proposition 3.5.6. *Every symmetric monoidal homomorphism $H \in \mathsf{SymBicat}(\mathsf{F}(G), \mathsf{M})$ is equivalent to one in the image of the canonical homomorphism:*

$$j : G(\mathsf{M}) \to \mathsf{SymBicat}(\mathsf{F}(G), \mathsf{M}).$$

Proof. Given $H = (H, \chi, \iota, \omega, \gamma, \delta u)$ we construct an object $\mathrm{res}(H)$ of $G(\mathsf{M})$ together with a canonical equivalence transformation θ between H and $j \circ \mathrm{res}(H)$. The object $\mathrm{res}\, H$ is essentially the *restriction* of H to the generating sets G_0, G_1, and G_2. An object of $G(\mathsf{M})$ consists of three maps, and the first of these is precisely given by restricting H to $G_0 \subseteq \mathrm{ob}\ \mathsf{F}(G)$. This gives a map:

$$\mathrm{res}\, H : G_0 \to \mathsf{M}_0.$$

By our previous considerations, there is a canonical extension of $\operatorname{res} H$ to $BW(G_0)$, which usually will not agree with H. Thus we cannot simply take $\operatorname{res} H$ to be the actual restriction of H to G_1 and G_2; the sources and targets won't match up correctly, as required in Definition 3.5.5. However there is a canonical 1-morphism in M, $\theta_w : H(w) \to \operatorname{res} H(w)$, which is defined inductively as follows. On the elements $a \in G_0$, $\theta_a = I_a$ the identity 1-morphism, and $\theta_1 = \iota^* : H(1) \to 1$. This defines θ for words of length less of length one. For longer binary words, θ is defined inductively as the composition:

$$\theta_{u \otimes v} : H(u \otimes v) \xrightarrow{\chi^*} H(u) \otimes H(v) \xrightarrow{\theta_u \otimes \theta_v} \operatorname{res} H(u) \otimes \operatorname{res} H(v) = \operatorname{res} H(u \otimes v).$$

Similarly $\theta_u^* : \operatorname{res} H(u) \to H(u)$ is defined using ι and χ in place of ι^* and χ^*. We define $\operatorname{res} H : G_1 \to \mathsf{M}_1$ by setting $\operatorname{res} H(f)$ to be the composition $\operatorname{res} H(f) = [\theta_{t(f)} \circ H(f)] \circ \theta^*_{s(f)}$. We also define $\operatorname{res} H : G - 2 \to \mathsf{M}_2$ by setting $\operatorname{res} H(\alpha) = [id_{\theta_{t(f)}} * H(\alpha)] * id_{\theta^*_{s(f)}}$, i.e. by whiskering. The data $\operatorname{res} H : (G_0, G_1, G_1) \to (\mathsf{M}_0, \mathsf{M}_1, \mathsf{M}_2)$ now defines an object of $G(\mathsf{M})$.

The components, θ_u, are part of what will be a symmetric monoidal transformation $\theta : H \to j(\operatorname{res} H)$. We must define θ on the 1-morphisms of $\mathsf{F}(G)$. This is again done inductively. For any binary sentence $s : u \to v$, we must have a 2-morphism $\theta_s : \operatorname{res} H(s) \circ \theta_u \to \theta_v \circ H(s)$. For $f \in G_1$, we have $\operatorname{res} H(s) = [\theta_v \circ H(f)] \circ \theta_u^*$, thus we may take θ_s to be the obvious composition of associators, unitors, and the adjunction 2-isomorphism $\theta_u^* \circ \theta_u \to I_u$. Similarly, we define θ on I_u by the pasting diagram,

$$\begin{tikzcd}
H(u) \arrow[rr, "\theta_u"] \arrow[d, "H(I_u)"'] \arrow[d, bend left=60, "I_{Hu}"] \arrow[d, phantom, shift left=2, "\overset{\phi^H}{\Leftarrow}"] & \arrow[d, phantom, "\Leftarrow"] & \operatorname{res} H(u) \arrow[d, "I_u = \operatorname{res} H(I_u)"] \\
H(u) \arrow[rr, "\theta_u"'] & & \operatorname{res} H(u)
\end{tikzcd}$$

Here the unlabeled 2-morphism is the obvious composition of unitors. With this definition θ automatically satisfies the second commutative diagram in Figure B.1.

For α, ℓ, r, and β we can use similar pasting diagrams, involving the coherence morphisms $\omega, \gamma, \delta, u$ and their mates. For example θ_ℓ is defined by the pasting diagram:

$$\begin{tikzcd}
H(1\otimes a) \arrow[r, "\chi^*"] \arrow[dd, "H(\ell)"'] \arrow[ddr, phantom, "\Leftarrow \gamma^*"] & H(1)\otimes H(a) \arrow[rr, "\iota^*\otimes\theta_a"] \arrow[dr, "\iota^*\otimes I"'] & \arrow[d, phantom, "\Downarrow"] & 1\otimes \operatorname{res} H(a) = \operatorname{res} H(1\otimes a) \arrow[dd, "\ell = \operatorname{res} H(\ell)"] \\
& & 1\otimes H(a) \arrow[ur, "I\otimes\theta_a"'] \arrow[dll, "\ell"'] \arrow[d, phantom, "\Leftarrow \ell"] & \\
H(a) \arrow[rrr, "\theta_a"'] & & & \operatorname{res} H(a)
\end{tikzcd}$$

The pasting diagrams for α, r, and β are similar and use mates of ω, δ and u, respectively.

This defines θ on 1-morphisms of length and width one. We extend it to all binary sentence by induction. If $f \circ g$ is a binary sentence in G_1, then we define $\theta_{g\circ f}$ to be the unique 2-morphism filling the following diagram:

$$\begin{tikzcd}
(\operatorname{res} Hg \circ \operatorname{res} Hf)\circ\theta_a \arrow[rr, dashed, "\theta_{gf}"] \arrow[d, "a^{\mathsf{M}}"] & & \theta_c \circ H(g\circ f) \\
\operatorname{res} Hg \circ (\operatorname{res} Hf \circ \theta_a) \arrow[d, "id_{\operatorname{res} Hg} * \theta_f"'] & & \theta_c \circ (Hg \circ Hf) \arrow[u, "id_{\theta_c} * \phi_{g,f}"'] \\
\operatorname{res} Hg \circ (\theta_b \circ Hf) \arrow[dr, "(a^{\mathsf{M}})^{-1}"'] & & (\theta_c \circ Hg)\circ Hf \arrow[u, "a^{\mathsf{M}}"'] \\
& (\operatorname{res} Hg \circ \theta_b)\circ Hf \arrow[ur, "\theta_g * id_{Hf}"'] &
\end{tikzcd}$$

It is instructive to compare this with Definition B.1.6 of transformations of bicategories. Defining θ on compositions as above ensures that θ satisfies the first diagram of Figure B.1. Finally, we must inductively define θ on tensor products $f \otimes g$. This is given by the following pasting diagram:

$$\begin{tikzcd}
H(a\otimes b) \arrow[r, "\chi^*"] \arrow[d, "H(f\otimes g)"'] \arrow[dr, phantom, "\overset{\chi^*}{\Leftarrow}"] & H(a)\otimes H(b) \arrow[r, "\theta_a\otimes\theta_b"] \arrow[d, "H(f)\otimes H(g)"] \arrow[dr, phantom, "\overset{\theta_f\otimes\theta_g}{\Leftarrow}"] & \operatorname{res} H(a)\otimes \operatorname{res} H(b) = \operatorname{res} H(a\otimes b) \arrow[d, "\operatorname{res} H(f\otimes g)"] \\
H(a'\otimes b') \arrow[r, "\chi^*"'] & H(a')\otimes H(b') \arrow[r, "\theta_{a'}\otimes\theta_{b'}"'] & \operatorname{res} H(a')\otimes \operatorname{res} H(b') = \operatorname{res} H(a'\otimes b')
\end{tikzcd}$$

With these definitions θ becomes a transformation $\theta : H \to \mathrm{res}\, H$, and similarly $\theta^* : \mathrm{res}\, H \to H$ becomes an inverse transformation. We will now promote θ to be a symmetric monoidal transformation by equipping it with modifications Π and M. For M, we must have a 2-morphism which fills the diagram:

$$\begin{tikzcd}
1 \arrow[rr, "I"] \arrow[dr, bend right, "\iota"'] & \arrow[d, phantom, "\Uparrow M"] & \mathrm{res}\, H(1) \\
& H(1) \arrow[ur, bend right, "\theta_1 = \iota^*"'] &
\end{tikzcd}$$

The obvious choice is the adjunction 2-morphism for ι, and this is what we will take. Similarly we use the adjuction 2-morphisms for χ to define Π, which is given precisely by the following pasting diagram:

$$\begin{tikzcd}
& H(a\otimes b) \arrow[dr, "\chi^*"] & & \\
Ha\otimes Hb \arrow[ur, "\chi"] \arrow[rr, bend right, "I"'] \arrow[d, "\theta\otimes I"'] \arrow[r, phantom, "\Uparrow \eta"] & {} & Ha\otimes Hb \arrow[r, "\theta_a\otimes\theta_b"] & \mathrm{res}\, Ha\otimes \mathrm{res}\, Hb \\
\mathrm{res}\, Ha\otimes Hb \arrow[rrr, bend right, "I\otimes\theta"'] \arrow[urr, phantom, "\Uparrow"] & & & \mathrm{res}\, Ha\otimes\mathrm{res}\, Hb \arrow[u, "I"']
\end{tikzcd}$$

Here η is an adjunction 2-morphism for χ and the unlabeled 2-morphism is the canonical one given by the homomorphism structures of $\otimes$.

With these choices of Π and M, θ becomes a symmetric monoidal transformation. The necessary coherence diagrams commute precisely because of our choices for θ and the fact that ω, γ, δ, and u come from a symmetric monoidal homomorphism. A similar statement holds for θ^*, which is easily seen to be and inverse equivalence transformation. $\square$

In the proof of the last proposition, we constructed an object $\mathrm{res}\, H \in G(\mathsf{M})$ for every homomorphism $H \in \mathsf{SymBicat}(\mathsf{F}(G), \mathsf{M})$. In fact this fits into homomorphism:

$$\mathrm{res} : \mathsf{SymBicat}(\mathsf{F}(G), \mathsf{M}) \to G(\mathsf{M}),$$

which satisfies $\mathrm{res} \circ j = id_{G_{\mathsf{M}}}$, so that we may regard j as an inclusion. This follows from the following proposition.

Proposition 3.5.7. *The homomorphism* $j : G(\mathsf{M}) \to \mathsf{SymBicat}(\mathsf{F}(G), \mathsf{M})$ *is an equivalence of bicategories.*

Proof. Proposition 3.5.6 asserts that the homomorphism j essentially surjective on objects. By Theorem B.1.15 (see also Theorem 3.4.10), we must show that it is also fully-faithful on 2-morphisms and essentially full on 1-morphisms.

Recall that the objects and 1-morphisms in the image of j are strict, i.e. the higher coherence data $(\chi, \iota, \omega, \gamma, \delta, u)$ and (M, Π) are trivial. Let H and $\overline{H}$ be two homomorphisms in the image of j and let θ and $\tilde{\theta}$ be two transformations between these, also in the image of j. Then axioms (BMBM1) and (BMBM2) of Definition 3.2.4 ensure that any modification $m : \theta \to \tilde{\theta}$ is in fact determined by its restriction to $G_0 \subseteq \mathsf{F}(G)$, and agrees with j applied to this restriction. Thus j is fully-faithful on 2-morphisms.

It remains to show, given H and $\overline{H}$ in the image of j, that any symmetric monoidal transformation (θ, Π, M) between them is equivalent to one in the image of j. Since H and $\overline{H}$ are in the image of j, the restriction of θ to G_0 and G_1 is an actual 1-morphism of $G(\mathsf{M})$ (between the corresponding objects of $G(\mathsf{M})$). Call this $\operatorname{res}\theta$. We will show $j \operatorname{res}\theta$ is equivalent to θ.

We must construct an invertible symmetric monoidal modification $m : j \operatorname{res}\theta \to \theta$. This consists of 2-morphisms,

$$m_u : j \operatorname{res}\theta(u) \to \theta(u)$$

for each object $u \in BW(G_0)$, which must satisfy certain coherence conditions. In fact Π and M provide a canonical choice form_u.

Recall that $j \operatorname{res}\theta(u)$ is an iterated binary tensor product of θ restricted to G_0. Since H and $\overline{H}$ are strict, Π provides 2-morphisms $\theta_a \otimes \theta_b \to \theta_{a\otimes b}$ for any pair of objects $a, b \in \mathbb{F}(G)$, and M provides a 2-morphism $\theta_1 \to I_1$. Thus given any binary word u in G_0 we may apply a sequence of these 2-morphisms Π and M to $j \operatorname{res}\theta(u)$ to obtain $\theta(u)$. Any two choices of sequence result in the same 2-morphism, as can be proven by appealing to Lemma 3.4.3, (an argument we have demonstrated several times before). The fact that this is an (invertible) symmetric monoidal modification then follows readily from the coherence axioms of Π and M. $\square$

The interpretation of the above proposition is that the symmetric monoidal bicategory $\mathsf{F}(G)$ is the *free symmetric monoidal bicategory generated by* G. The bicategory of

symmetric monoidal homomorphism from $\mathsf{F}(G)$ into any other symmetric monoidal bicategory, M, is equivalent to the bicategory of G-data in M.

3.6 Relations for Symmetric Monoidal Bicategories

In this work we only consider relations which occur at the level of 2-morphisms, although more general constructions do exist. Let M be a symmetric monoidal bicategory equipped with a set $\mathcal{R}$ of pairs of 2-morphism $(\alpha, \beta) \in \mathcal{R}$ such that the sources and targets of α and β agree. $\mathcal{R}$ is called the set of *generating relations*. Let $\sim$ denote the finest equivalence relation on the 2-morphisms of M such that:

1. $\alpha \sim \beta$ if $(\alpha, \beta) \in \mathcal{R}$,
2. $f \circ \alpha \circ g \sim f \circ \beta \circ g$ whenever these compositions exist and $\alpha \sim \beta$,
3. $\alpha * g \sim \beta * g$ and $f * \alpha \sim f * \beta$ whenever these compositions exist and $\alpha \sim \beta$,
4. $\alpha \otimes g \sim \beta \otimes g$ and $f \otimes \alpha \sim f \otimes \beta$ whenever $\alpha \sim \beta$.

Let $x, y \in \mathsf{M}$ be objects and $f, g : x \to y$ be 1-morphisms in M. Denote by $(\mathsf{M}/\mathcal{R})(x,y)(f,g)$ the set $\mathsf{M}(x,y)(f,g)/\sim$, and by $(\mathsf{M}/\mathcal{R})_2$ the totality $\mathsf{M}_2/\sim$. We define the following partial operations on the sets $(\mathsf{M}/\mathcal{R})(x,y)(f,g)$:

1. vertical composition: $[\alpha] \circ [\beta] := [\alpha \circ \beta]$,
2. horizontal composition: $[\alpha] * [\beta] := [\alpha * \beta]$,
3. tensor product: $[\alpha] \otimes [\beta] := [\alpha \otimes \beta]$.

These operations are only given when the corresponding operation is defined on representatives. The above required properties of $\sim$ ensure that these operations are well defined. The vertical composition maps $(\mathsf{M}/\mathcal{R})(x,y)$ into a category with the same objects as $\mathsf{M}(x,y)$. Let $\mathsf{M}/\mathcal{R}$ denote the collection of the objects of M together with these categories $(\mathsf{M}/\mathcal{R})(x,y)$. We equip $\mathsf{M}/\mathcal{R}$ with associator and unitor natural isomorphisms induced by their images under the evident quotient map $\mathsf{M}_2 \to (\mathsf{M}/\mathcal{R})_2$. In this way $\mathsf{M}/\mathcal{R}$ becomes a bicategory. Similarly $\otimes$, together with the images of the unit and coherence data of M, equip $\mathsf{M}/\mathcal{R}$ with the structure of a symmetric monoidal bicategory. The quotient map $\mathsf{M} \to \mathsf{M}/\mathcal{R}$ is a strict symmetric monoidal (strict) homomorphism.

Proposition 3.6.1. *The quotient homomorphism* $\mathsf{M} \to \mathsf{M}/\mathcal{R}$ *satisfies the following universal property:*

$$\mathsf{SymBicat}(\mathsf{M}/\mathcal{R}, \mathsf{S}) \cong \mathsf{SymBicat}_{\mathcal{R}}(\mathsf{M}, \mathsf{S})$$

for all symmetric monoidal bicategories S. *Here* $\mathsf{SymBicat}_{\mathcal{R}}$ *denotes the full sub-bicategory consisting of those symmetric monoidal homomorphisms* H, *such that* $H(\alpha) = H(\beta)$ *whenever* $(\alpha, \beta) \in \mathcal{R}$.

Proof. Pre-composition with $\mathsf{M} \to \mathsf{M}/\mathcal{R}$ clearly lands in $\mathsf{SymBicat}_{\mathcal{R}}(\mathsf{M}, \mathsf{S})$. Given a symmetric monoidal homomorphism $H : \mathsf{M} \to \mathsf{S}$ in $\mathsf{SymBicat}_{\mathcal{R}}(\mathsf{M}, \mathsf{S})$, the properties of $\sim$ and H ensure that $H(\alpha) = H(\beta)$ whenever $\alpha \sim \beta$. Hence there is, in fact, a unique extension of H to $\mathsf{M}/\mathcal{R}$. It is also follows, since $\mathsf{M}/\mathcal{R}$ has the same objects and 1-morphisms as M, that symmetric monoidal transformations and modifications extend uniquely as well, so that we indeed have an isomorphism of bicategories. $\square$

Chapter 4

The Classification of 2-Dimensional Extended Topological Field Theories

In this chapter we assemble the results of the previous two chapters to prove our main classification theorems. The majority of the results of this chapter are entirely new. There are four primary goals for this chapter. First we must introduce the symmetric monoidal bicategory of bordisms with structure. Previous attempts at defining the bordism bicategory have been made in [KL01] and [Mor07], but neither of these is adequate. First, they essentially ignore the *symmetric monoidal* structure on the bordism bicategory, which is essential to define extended topological field theories. This is perhaps understandable, since they did not have available the results of Chapter 3.

Additionally, neither [KL01] nor [Mor07], discuss the issue of gluing bordisms in sufficient detail to adapt their definition to a bicategory of bordisms with structure. The main issue is that when one glues two smooth manifolds along a common boundary, the resulting space, $W \cup_Y W'$, which is a topological manifold, does not have a unique smooth structure extending those on W and W'. While any two choices of such smooth structure result in diffeomorphic manifolds, this diffeomorphism is not *canonical*. In the 1-categorical bordism category, this issue is easily swept under the rug. But in the higher categorical setting it must be dealt with. Kerler-Lyubashenko [KL01] and Morton [Mor07], narrowly avoid this issue.

In Section 4.1 we discuss these gluing issues at length and give a preliminary, naive version of the d-dimensional bordism bicategory as a symmetric monoidal bicategory. In Section 4.2 we improve upon this definition and give the definition of the symmetric monoidal bicategory of bordisms with $\mathcal{F}$-structures. The allowed structures are very general and include structures which can be described by sheaves, such as orientations and metrics, or even stacks, such as super manifold structures, spin strictures, or principle bundles.

In Section 4.3 we address our second primary goal, which is to prove our main classification theorem (Theorem 4.3.1) for 2-dimensional extended unoriented topological field theories. This theorem gives a generators and relations presentation for the extended 2-dimensional unoriented bordism bicategory as a symmetric monoidal bicategory. Using the results of Chapeters 2 and 3, the proof is relatively straightforward. This effectively classifies extended 2-dimensional unoriented topological field theories with *any* target symmetric monoidal bicategory. This is the first time such a result has been rigorously proven in the literature.

In Section 4.4 we prove our third primary goal, and show how the techniques developed in this dissertation can be generalized and adapted to classify bordism bicategories with structure. We demonstrate this by proving the analogous classification theorem (Theorem 4.4.1) for the 2-dimensional *oriented* extended bordism bicategory.

Finally, in Section 4.6, we achieve our fourth primary goal. We use the classification results of previous section to classify extended topological field theories with values in the symmetric monoidal bicategory of algebras, bimodules, and intertwiners. In particular, over a perfect field we prove that extended oriented topological field theories are the same as semi-simple (non-commutative) Frobenius algebras and that in the unoriented case they are the same as semi-simple Frobenius $*$-algebras.

4.1 The Bordism Bicategory I: Gluing

Atiyah's axiomatic definition of topological quantum field theories hinges upon the existence of the bordism category. The objects and morphism of this category are clear. The objects (for the d-dimensional unoriented category) are closed $(d-1)$-dimensional manifolds and the morphisms from W_0 to W_1 are d-dimensional (1-)bordisms from W_0 to W_1, i.e. compact d-manifolds S equipped with a decomposition and isomorphism of their boundaries $\partial S = \partial_{\text{in}} S \sqcup \partial_{\text{out}} S \cong W_0 \sqcup W_1$. These bordisms are taken up to isomorphism of

bordism, i.e. isomorphisms $S \cong S'$, such that the induced map,

$$W_0 \sqcup W_1 \cong \partial_{\text{in}} S \sqcup \partial_{\text{out}} S = \partial S \cong \partial S' = \partial_{\text{in}} S' \sqcup \partial_{\text{out}} S' \cong W_0 \sqcup W_1,$$

is the identity.

In the purely topological setting (i.e. where these manifolds are *topological* manifolds) composition is clear and is given simply by gluing bordisms in the obvious way. In the smooth setting, gluing is mildly more complicated. A smooth structure on the topological manifold $S \cup_W S'$ is not uniquely determined by smooth structures on S and S'. In addition one needs to specify a tubular neighborhood $W \times \mathbb{R} \hookrightarrow S \cup_W S'$ which is smooth when restricted to the collars $W_+ = W \times [0, \infty) \hookrightarrow S$ and $W_- = W \times (-\infty, 0] \hookrightarrow S'$. Different choices of collars generally induce different smooth structures on $S \cup_W S'$. Nevertheless, the resulting smooth manifolds are diffeomorphic, but by a non-canonical diffeomorphism (as we shall see shortly). This is sufficient to provided the bordism category with a well-defined composition and one easily verifies that it is, indeed, a category.

In the higher categorical setting these mild difficulties are only amplified. In the bicategorical case what one is after is a bicategory whose objects are closed $(d-2)$-manifolds, Y, whose 1-morphism are compact $(d-1)$-dimensional 1-bordisms, W, between these and whose 2-morphisms are compact bordisms, S, between the 1-bordisms. Such a setup almost invariably requires the use of manifolds with corners. More seriously, in such a bicategory one must be able to glue the 1-morphisms (1-bordisms) horizontally. This is problematic because, as we have just seen, this gluing is not well defined on the nose, only up to *non-canonical* diffeomorphism. One may be tempted to take diffeomorphism classes 1-bordisms as the 1-morphisms, but such an approach is doomed to failure since one will be unable to construct a acceptable notion of 2-bordism.

Alternatively one may try to skirt this issue by using the axiom of choice to choose for each pair of composable 1-bordisms, W and W', a smooth composite $W \circ W'$ (as suggested in [Mor07, §5.1]) or equivalently by choosing a skeleton (or finite set of representatives) for each diffeomorphism class of 1-bordism (as done in [KL01]). Both of these approaches result in well defined composition functors for pairs of composable 1-bordisms, and moreover construct a natural diffeomorphism:

$$a : (W \circ W') \circ W'' \to W \circ (W' \circ W'').$$

However this natural diffeomorphism will *fail* to satisfy the pentagon axiom! This problem

is narrowly avoided in [KL01] and [Mor07], but for subtle reasons.

There doesn't appear to be an adequate discussion elucidating these issues in the literature and so, in this section, we begin by discussing gluing of bordisms and provide a solution to the above problem. This also provides us an opportunity to freshen the mind of the reader and to fix terminology. In the next section we will give an alternative solution to this problem which is more compatible with equipping our bordisms with additional structures.

Let $\mathbb{R}_+ = \{y \in \mathbb{R} \mid y \geq 0\} = [0, \infty)$ and let X be a topological m-manifold with boundary. Let $x \in X$. A *chart at x* (of *index p*) is a continuous map $\varphi : U \to \mathbb{R}^m_+$, where U is an open subset of X containing x, φ is a homeomorphism onto its image, and in which $\varphi(x)$ has p-many coordinates which take the value zero. Two charts $\varphi : U \to \mathbb{R}^m_+$ and $\varphi' : U' \to \mathbb{R}^m_+$ are *compatible* if the composite,

$$\varphi' \circ \varphi^{-1} : V \to \mathbb{R}^m_+$$

is a diffeomorphism onto its image, where $V = \varphi(U \cap U')$. Here if $S \subset \mathbb{R}^m_+$ is a subset, then $f : S \to \mathbb{R}^m_+$ is *smooth* if there exists an open neighborhood $W \subset \mathbb{R}^m$ of S and a smooth extension of f to W. Note that if φ and φ' are two charts at x which are compatible, then the index of φ coincides with the index of φ'.

If X is a topological manifold with boundary then an *atlas* for X is a compatible family of charts, containing a chart at x every point $x \in X$. Given an atlas for X, the *index* of a point $x \in X$ is the index of any chart at x. Atlases are ordered by inclusion. A *manifold with corners* is a topological manifold with boundary equipped with a maximal atlas. Given an atlas $\mathcal{A}$ for X, there exists a unique maximal atlas containing $\mathcal{A}$.

A submanifold of a manifold with corners X is a manifold with corners Y with an embedding $i : Y \hookrightarrow X$, such that in any charts at y for Y and at $i(y)$ for X, i is smooth. Let Y be a submanifold of X such that $Y \subset \partial X$. Then a *collar* of Y is a map $f_Y : U_Y \to Y \times \mathbb{R}_+$ such that U_Y is an open neighborhood of Y in X, $f_Y(y) = (y, 0)$ for all $y \in Y$, and f_Y is a diffeomorphism of manifolds with corners. Not every submanifold $Y \subset \partial X$ admits a collar, as the example in Figure 4.1 demonstrates.

A *connected face* of a manifold with corners X is the closure of a component of $\{x \in X \mid \text{ index}(x) = 1\}$. A manifold with corners is a *manifold with faces* if each $x \in X$ belongs to index(x) different connected faces. For such a manifold a *face* is a disjoint union of connected faces, and is itself a manifold with faces (and hence a manifold with corners).

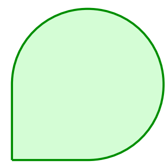


Figure 4.1: A manifold with corners without a collar.

If X is a manifold with faces, then so is $\mathbb{R}_+ \times X$.

Proposition 4.1.1. *Let X and X' be manifolds with faces and Y a manifold with corners, together with inclusions $f : Y \to X$ and $f' : Y \to X'$ realizing Y as a face of each. Then $X \cup_Y X'$ is a topological manifold with boundary. If in addition we are given collars $f_+ : Y \times \mathbb{R}_+ \to X$ and $f_- : Y \times \mathbb{R}_+ \to X'$, there exists a canonical smooth structure on $X \cup_Y X'$ which is compatible with the smooth structures on X and X'.*

Proof. The property that $X \cup_Y X'$ is a topological manifold with boundary is a completely local question and may be readily checked. To provide $X \cup_Y X'$ with the structure of a manifold with corners we must provide a maximal smooth atlas. It is sufficient to provide a covering by open smooth manifolds with corners, such that the transition functions are smooth on overlaps. Let $\breve{X}$ denote $X \setminus Y$ and similarly let $\breve{X}'$ denote $X' \setminus Y$. Then $\breve{X}$, $\breve{X}'$, and $\mathbb{R} \times Y$ provide such a cover, where the first two maps to $X \cup_Y X'$ are the obvious inclusions and the third, from $\mathbb{R} \times Y$, is given by the union of the collars f_+ and f_-. □

Proposition 4.1.2. *Let B be a closed $(d-1)$ manifold and let S_0 and S_1 be compact d-manifold. Let $B \to \partial S_0$ and $B \to \partial S_1$ be diffeomorphisms of B with components of the boundaries of S_0 and S_1. Then the push-out,*

$$\begin{tikzcd} B \arrow[r] \arrow[d] & S_1 \arrow[d] \\ S_0 \arrow[r] \arrow[ur, phantom, "\lrcorner", very near end] & S \end{tikzcd}$$

exists in the category of smooth manifolds if and only if $B = \emptyset$.

Proof. Suppose that the pushout S exists. We can calculate the smooth functions on S via the universal property of the pushout. We see that they consist of precisely those pairs of smooth functions on S_0 and S_1 which agree on B. In particular this determines

a topology on S and we see that as a topological space it coincides with the topological pushout $S_0 \cup_B S_1$. If B is not the emptyset, then there is no smooth manifold for which these are the smooth functions, which can be seen by considering a neighborhood in S of any point in B. □

Corollary 4.1.3. *There is no subcategory of smooth manifolds with finite pushouts containing B, S_0, and S_1 as above, with $B \neq \emptyset$ which contains the automorphisms of $S_0 \circ_B S_1$ which fix B and are the identity outside a neighborhood of B.*

The next lemma and theorem are taken from [Mun66, §6], the proofs of which are readily adapted to cover the case of manifolds with faces. We will delay the proof of the lemma until after that for the theorem.

Lemma 4.1.4. *Let M be a compact manifold with faces and N an (open) neighborhood of $M \times \{0\}$ in $M \times \mathbb{R}_+$. Let $f : N \hookrightarrow M \times \mathbb{R}_+$ be smooth and equal to the identity on $M \times \{0\}$. Then there exists a diffeomorphism $\tilde{f} : N \to f(N)$ such that*

1. *$\tilde{f} = f$ in a neighborhood of the complement of N,*
2. *$\tilde{f} = id$ in a neighborhood of $M \times \{0\}$.*

Theorem 4.1.5. *Let X and X' be manifolds with faces and Y a manifold with corners, together with inclusions $f : Y \to X$ and $f' : Y \to X'$ realizing Y as a face of each. Suppose that Y admits collars $f_+ : Y \times \mathbb{R}_+ \to X$ and $f_- : Y \times \mathbb{R}_+ \to X'$, but that these are not specified. Then the glued manifold with faces, $X \cup_Y X'$, is unique up to diffeomorphism fixing Y, and which coincides with the identity map outside a neighborhood of Y.*

Proof. Let $P = (f_+, f_-)$ and $P' = (f'_+, f'_-)$ be two choices of pairs of collars for Y. Then P is equivalent to a map $P : Y \times \mathbb{R} \to X \cup_Y X'$which is a homeomorphism onto its image, V, and smooth on the complement of $Y \times \{0\}$. Similarly, $P' : Y \times \mathbb{R} \to X \cup_Y X'$ is a map which a is a homeomorphism onto its image, V', and smooth on the complement of $Y \times \{0\}$.

Let $U = P^{-1}(V \cap V') \subset Y \times \mathbb{R}$. Then the composite $(P')^{-1} \circ P$ is defined on U and is a homeomorhism onto $(P')^{-1} \circ P(U)$. Moreover it is the identity on $Y \times \{0\}$ and is smooth when restricted to $U_+ = U \cap Y \times [0, \infty)$ and $U_- = U \cap Y \times (-\infty, 0]$, separately.

By Lemma 4.1.4 there exists a homeomorphism $g : U \to (P')^{-1}P(U)$ such that

1. g equals $(P')^{-1}P$ near the boundary of U,

2. g is a diffeomorphism when restricted to U_+ and U_-,

3. g coincides with the identity map in a neighborhood of $Y \times \{0\}$.

Hence $P'gP^{-1}$ is defined on the neighborhood $V \cap V'$ of Y in $X \cup_Y X'$ and coincides with the identity map near the boundary of this neighborhood. Extending by the identity map yields a homoeomorphism f of $X \cup_Y X'$. One may readily check that f is a diffeomorphism. □

Proof of Lemma 4.1.4. There exists a smooth function $\beta : M \to [0, 1)$ such that $N' = \{(x, t) \in M \times \mathbb{R}_+ \mid t \leq \beta(x)\} \subseteq N$. Proving the lemma for N' proves it for N, hence without loss of generality we may assume that $N = N'$.

Let $f(x, t) = (X(x, t), T(x, t))$. Then $\frac{\partial T}{\partial t}(x, 0) > 0$. Hence we may choose $\varepsilon : M \to (0, 1)$, such that

1. $\frac{\partial T}{\partial t}(x, t) > 0$ for all $0 \leq t \leq \varepsilon(x)$,

2. $\varepsilon(x)|\frac{\partial T}{\partial t}(x, t)| < 1$ for all $0 \leq t \leq \beta(x)$.

Choose a smooth monotone function $\alpha : \mathbb{R} \to \mathbb{R}$, such that $\alpha(t) = 0$ for all $t \leq \frac{1}{3}$ and $\alpha(t) = 1$ for all $t \geq \frac{2}{3}$. Define the function on N,

$$\psi(x, t) = (1 - \alpha(\frac{t}{\beta(x)}))\varepsilon(x)t + \alpha(\frac{t}{\beta(x)})t.$$

Then $g(x, t) = (x, \psi(x, t))$ defines a diffeomorphism $g : N \to N$.

Setting $f_1 = f \circ g : N \to f(N)$, we see that f_1 is also a diffeomorphism. Let $f_1(x, t) = (X_1(x, t), T_1(x, t))$. Define the function on N,

$$\phi(x, t) = \alpha(\frac{2t}{\beta(x)})t + [1 - \alpha(\frac{2t}{\beta(x)})]T_1(x, t).$$

Then $h(x, t) = (x, \phi(x, t))$ defines a diffeomorphism $h : N \to N$. Set $f_2 = f_1 \circ h^{-1}$, and let $f_2(x, t) = (X_2(x, t), T_2(x, t))$.

We have that $T_2 \equiv t$ on a neighborhood L of $M \times \{0\}$, and since M is compact, there exists a $\delta > 0$ such that $M \times [0, \delta] \subseteq L$. Finally we let

$$X_3(x, t) = X_2(x, \alpha(\frac{t}{\delta})t),$$

and define $\tilde{f}(x, t) = (X_3(x, t), T_2(x, t))$. One may check that $\tilde{f}$ has the desired properties. □

Remark 4.1.6. The diffeomorphism constructed in Lemma 4.1.4 depends on the choice of functions ε, β and α. Any other choice of such data $(\tilde{\varepsilon}, \tilde{\beta}, \tilde{\alpha})$ is isotopic to the original choice of data through valid choices of this data. In fact the space of such data is a contractible space. Tracing through the proof of Lemma 4.1.4 and Theorem 4.1.5 we see that this implies that, while the diffeomorphism constructed in the proof of Theorem 4.1.5 is not canonical, it does have a *canonical isotopy class.* $\diamond$

Following [Jän68, Lau00, Mor07] we will define our higher bordisms as manifolds with faces equipped with additional structure.

Definition 4.1.7. An $\langle n \rangle$*-manifold* is a manfold with faces X together with an ordered n-tuple $(\partial_0 X, \ldots, \partial_{n-1} X)$ of faces of X which satisfy the following conditions:

1. $\partial_0 X \cup \cdots \cup \partial_{n-1} X = \partial X$, and
2. $\partial_i X \cap \partial_j X$ is a (possibly empty) face of $\partial_i X$ and $\partial_j X$ for all $i \neq j$.

$\diamond$

Let [1] denote the ordered set $[0,1]$, viewed as a category. Let $\langle n \rangle$ denote the category $[1]^{\times n}$. An $\langle n \rangle$-manifold induces an $\langle n \rangle$-diagram of spaces, i.e. a functor $X : \langle n \rangle \to \mathsf{Top}$. If X is an $\langle n \rangle$-manfold and $a = (a_0, \ldots, a_{n-1})$ is an object of $\langle n \rangle$, then

$$X(a) = \cap_{a_i \neq 0} \partial_i X$$

with morphisms being sent to the obvious inclusions. By convention $X(0, 0, \ldots, 0) = X$.

Example 4.1.8. $X = \mathbb{R}^m_+$ is a manifold with faces in the obvious manner. The faces consist of the m-coordinate hyperplanes. Order these hyperplanes $H_0, H_1, \ldots, H_m$. Then X becomes an $\langle m \rangle$-manifold where $\partial_i X = H_i$. Moreover, if e_i, $0 \leq i \leq m-1$ denote the standard basis vectors for $\mathbb{R}^m$, then

$$X(a_0, \ldots, a_{m-1}) = \mathrm{Span}\{e_i \mid a_i = 0\}.$$

$\diamond$

The combinatorics of $\langle n \rangle$-manifolds is particularly well suited to our purposes. In addition, the faces of $\langle n \rangle$-manifolds admit collars in the sense of the following Lemma.

Lemma 4.1.9 ([Lau00, Lemma 2.1.6]). *Any $\langle n\rangle$-manifold X admits and $\langle n\rangle$-diagram C of embeddings*

$$C(a < b) : \mathbb{R}^n_+(a') \times X(a) \hookrightarrow \mathbb{R}^n_+(b') \times X(b)$$

with the property that $C(a < b)$ restricted to $\mathbb{R}^n_+(b') \times X(a)$ is the inclusion map $id \times X(a < b)$. Here $a' = (1, \dots, 1) - a$ and $b' = (1, \dots, 1) - b$.

With these definitions at hand we are ready to give the first definition of the d-dimensional bordism bicategory $d\text{-}\mathsf{Cob}_2$. The objects are smooth closed $(d-2)$-dimensional manifolds Y. The 1-morphisms and 2-morphisms are 1-bordisms and isomorphism classes of 2-bordisms, which we define shortly.

Definition 4.1.10. Let Y_0 and Y_1 be smooth closed $(d-2)$-manifolds. A *1-bordism* from Y_0 to Y_1 is a smooth compact $(d-1)$-manifold with boundary, W, equipped with a decomposition and isomorphism of its boundary $\partial W = \partial_{\text{in}} W \sqcup \partial_{\text{out}} W \cong Y_0 \sqcup Y_1$. ⋄

Definition 4.1.11. Let Y_0 and Y_1 be smooth closed $(d-2)$-manifolds and let W_0 and W_1 be two 1-bordisms from Y_0 to Y_1. A *2-bordism* consists of a compact d-dimensional $\langle 2\rangle$-manifold, S, equipped with the following additional structures:

– A decomposition and isomorphism:

$$\partial_0 S = \partial_{0,\text{in}} S \sqcup \partial_{0,\text{out}} S \xrightarrow{g} W_0 \sqcup W_1.$$

– A decomposition and isomorphism:

$$\partial_1 S = \partial_{1,\text{in}} S \sqcup \partial_{1,\text{out}} S \xrightarrow{f} Y_0 \times I \sqcup Y_1 \times I.$$

These are required to induce isomorphisms

$$\begin{aligned} f^{-1}g :&\partial_{\text{in}} W_0 \sqcup \partial_{\text{out}} W_0 \to Y_0 \times \{0\} \sqcup Y_1 \times \{0\} \\ f^{-1}g :&\partial_{\text{in}} W_1 \sqcup \partial_{\text{out}} W_1 \to Y_0 \times \{1\} \sqcup Y_1 \times \{1\}, \end{aligned}$$

which coincide with the structure isomorphisms of W_0 and W_1.

Two 2-bordisms, S and S', are isomorphic if there is a diffeomorphism $h : S \to S'$, which restricts to diffeomorphisms,

$$\partial_{0,\mathrm{in}}S \to \partial_{0,\mathrm{in}}S'$$
$$\partial_{0,\mathrm{out}}S \to \partial_{0,\mathrm{out}}S'$$
$$\partial_{1,\mathrm{in}}S \to \partial_{1,\mathrm{in}}S'$$
$$\partial_{1,\mathrm{out}}S \to \partial_{1,\mathrm{out}}S'$$

and such that $f' \circ h = f$ and $g' \circ h = g$. ◇

Thus a 2-bordism consists of a diagram of manifolds as in Figure 4.2. There are several equivalence relations on diffeomorphism that will be useful for us.

$$\begin{tikzcd}
 & & \partial_{\mathrm{out}}W_0 & \subset & W_0 & \supset & \partial_{\mathrm{in}}W_0 & & \\
 & Y_1\times\{0\} & \partial_{\mathrm{out}}\partial_{0,\mathrm{in}}S \arrow[l, dashed] \arrow[u, dashed] & \subset & \partial_{0,\mathrm{in}}S \arrow[u, "g_{\mathrm{in}}"] & \supset & \partial_{\mathrm{in}}\partial_{0,\mathrm{in}}S \arrow[u, dashed] \arrow[r, dashed] & Y_0\times\{0\} & \\
 & \cap & \cap & & \cap & & \cap & \cap & \\
Y_1 \arrow[uur] \arrow[ddr] & Y_1\times I & \partial_{1,\mathrm{out}}S \arrow[l, "f_{\mathrm{out}}"'] & \subset & S & \supset & \partial_{1,\mathrm{in}}S \arrow[r, "f_{\mathrm{in}}"] & Y_0\times I & Y_0 \arrow[uul] \arrow[ddl] \\
 & \cup & \cup & & \cup & & \cup & \cup & \\
 & Y_1\times\{1\} & \partial_{\mathrm{out}}\partial_{1,\mathrm{out}}S \arrow[l, dashed] \arrow[d, dashed] & \subset & \partial_{0,\mathrm{out}}S \arrow[d, "g_{\mathrm{out}}"] & \supset & \partial_{\mathrm{in}}\partial_{0,\mathrm{out}}S \arrow[d, dashed] \arrow[r, dashed] & Y_0\times\{1\} & \\
 & & \partial_{\mathrm{out}}W_1 & \subset & W_1 & \supset & \partial_{\mathrm{in}}W_1 & &
\end{tikzcd}$$

Figure 4.2: The structure of a 2-bordism

Definition 4.1.12. Let Y and Z be smooth manifolds (possibly with corners). Let $f_0, f_1 : Y \to Z$ be diffeomorphisms such that $f_0|_{\partial Y} = f_1|_{\partial Y}$. Then f_0 is *pseudo-isotopic* to f_1 if there exists a diffeomorphism $F : Y \times I \to Z \times I$ such that,

$$F|_{\partial Y \times I} = f_0|_{\partial Y} \times id : \partial Y \times I \to \partial Z \times I,$$

in a neighborhood of $Y \times \{0\}$, $F(x,t) = (f_0(x), t)$, and in a neighborhood of $Y \times \{1\}$, $F(x,t) = (f_1(x), t)$. ◇

Note that an isotopy is a particular kind of pseudo-isotopy, in which $F(y,t) = (f_t(y),t)$ is level preserving. The importance of pseudo-isotopy for the bordism bicategory arises from the following lemma.

Lemma 4.1.13. *Let (S,f,g) be a 2-bordism. If we replace f by a pseudo-isotopic map, then the resulting 2-bordism is in the same isomorphism class.*

Proof. Suppose that f and f' are psuedo-isotopic. This is equivalent to $f' \circ f^{-1}$ being psuedo-isotopic to the identity map on ∂S. Suppose that F is a diffeomorphism of $\partial S \times I$ realizing this pseudo-isotopy (which we may assume is the constant isotopy on $\partial_0 S$). We have that $F \cup id_S$ is a self-diffeomorphism of $(\partial S \times I) \cup S$ which is $f' \circ f^{-1}$ on the (∂_1) boundary. Choose an diffeomorphism $G : S \to (\partial S \times I) \cup S$. Then $h = G^{-1} \circ (F \cup Id_S) \circ G$ is an isomorphism between the 2-bordisms (S,f,g) and (S,f',g). See Figure 4.3. □

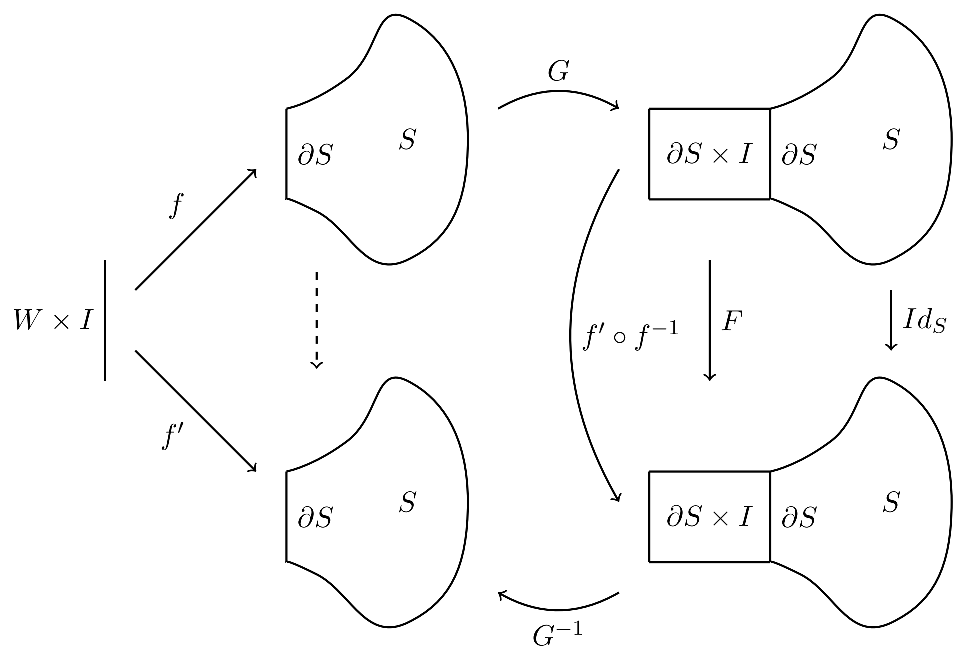


Figure 4.3: Pseudo-isotopy doesn't affect 2-morphisms

We can now define vertical composition of 2-bordisms. Given 1-bordisms W_0, W_1, W_2 from Y_0 to Y_1, and 2-bordisms S from W_0 to W_1 and S' from W_1 to W_2, we would like to define their vertical composition by choosing collars for W_1 in S and S', as per Lemma

4.1.9, and use these to glue S and S' together as in Proposition 4.1.1. The result is again a $\langle 2 \rangle$-manifold $S' \circ S$, which has a decomposition and isomorphism,

$$\partial_0(S' \circ S) = \partial_{0,\mathrm{in}} S \sqcup \partial_{0,\mathrm{out}} S' \xrightarrow{g_{\mathrm{in}} \sqcup g'_{\mathrm{out}}} W_0 \sqcup W_2.$$

However the second decomposition and isomorphism is problematic. It is as follows,

$$\partial_0(S' \circ S) = \partial_{1,\mathrm{in}} S \cup_{Y_0} \partial_{1,\mathrm{in}} S' \sqcup \partial_{1,\mathrm{out}} S \cup_{Y_1} \partial_{1,\mathrm{out}} S' \xrightarrow{f \cup f'} Y_0 \times (I \cup_{pt} I) \sqcup Y_1 \times (I \cup_{pt} I).$$

In order to make $S' \circ S$ into a 2-bordism, we must choose isomorphisms,

$$Y_i \times (I \cup_{pt} I) \cong Y_i \times I.$$

This can be done uniformly by choosing once and for all a diffeomorphism $I \cup_{pt} I \cong I$. Any two such choices are psuedo-isotopic. In this way, $S' \circ S$ becomes a 2-bordism from W_0 to W_2. There were several choices involved, but by Theorem 4.1.5, Remark 4.1.6, and Lemma 4.1.13 the isomorphism class of this 2-bordism is well defined and doesn't depend on the choices we've made. Moreover this vertical composition defines an operation which is associative on isomorphism classes of 2-bordisms, which again follows from Lemma 4.1.13.

Proposition 4.1.14. *Fix closed $(d-2)$-manifolds Y_0, Y_1. Then $\mathsf{Cob}_d(Y_0, Y_1)$ is a category, where the objects of $\mathsf{Cob}_d(Y_0, Y_1)$ consist of the 1-bordisms from from Y_0 to Y_1, the morphisms consist of isomorphism classes of 2-bordisms between these and composition is given by vertical composition of 2-bordism.*

Proof. The only detail left to be check is that there exist identity 2-bordisms. We leave it to the reader to verify that $W \times I$ gives such an identity. □

Horizontal composition of 2-bordisms and 1-bordisms is defined similarly, but we will need to use the axiom of choice. Given a 1-bordism W from Y_0 to Y_1 and a 1-bordism W' from Y_1 to Y_2, we would like to obtain a new 1-bordism, $W' \circ W$, from Y_0 to Y_1 by gluing together W and W'. Since our 1-bordisms are not taken up to isomorphism, the choices we make in defining this gluing are more important. Propositon 4.1.1 ensures that once we have chosen collars for Y_1, that we can form a canonical gluing. So in order to define the horizontal composition of 1-morphisms, we must use the axiom of choice to choose these collars for each 1-bordism.

There is no way to ensure that these choices are compatible, and so this gluing is associative only up to non-canonical isomorphism of 1-bordism. However, following Remark 4.1.6, there is a canonical isotopy class of diffeomorphisms and hence a canonical isomorphism class of 2-bordisms realizing the associativity of horizontal composition.

Horizontal composition of 2-morphisms can now be defined just as vertical composition. Given composible 2-bordisms

$$S \in \mathsf{Cob}_d(Y_0, Y_1)(W, W') \quad \text{and} \quad S' \in \mathsf{Cob}_d(Y_1, Y_2)(W'', W''')$$

we define $S' * S \in \mathsf{Cob}_d(Y_0, Y_2)(W'' \circ W, W''' \circ W')$ by choosing collars of $Y_1 \times I$ and gluing. These collars can be chosen to be compatible with our choices for W, W', W'' and W''' and the resulting isomorphism class of 2-bordism is well-defined. Similar considerations involving identities apply (identities are given by the 1-bordisms $Y_0 \times I$) and yield the following proposition.

Proposition 4.1.15. Cob_d *is a bicategory in which the objects are closed* $(d-2)$*-manifolds, the morphism categories are the categories* $\mathsf{Cob}_d(Y_0, Y_1)$*, and where composition is given by horizontal composition together with its canonical associators and unitors.*

Given a diffeomorphism $f : Y_0 \to Y_1$ of closed $(d-2)$-manifolds, we may promote it to a 1-bordism as follows. The $(d-1)$ manifold is $W = Y_0 \times I$, and the decomposition/isomorphism is given by,

$$\partial W = Y_0 \times \{0\} \sqcup Y_0 \times \{1\} \xrightarrow{id \sqcup f} Y_0 \sqcup Y_1.$$

This assignment respect composition, at least up to canonical (isomorphism class of) 2-bordism, and similarly isomorphisms of 1-bordisms may be turned into 2-bordisms. In particular we may use this to promote the symmetric monoidal structure on the categories of $(d-2)$-manifolds, 1-bordisms, and 2-bordisms given by disjoint union into a fully fledged symmetric monoidal structure on Cob_d.

Theorem 4.1.16. *With the structures given by disjoint union,* Cob_d *is a symmetric monoidal bicategory.*

Proof. Disjoint unions gives the categories of $(d-2)$-manifolds, 1-bordisms, and 2-bordisms the structure of symmetric monoidal categories. By turning these into 1-bordisms and

2-bordisms, as above, we obtain the data of a symmetric monoidal structure on the bicategory Cob_d. One only needs to verify that these satisfy the requisite axioms, which is straightforward, although tedious. □

4.2 The Bordism Bicategory II: Halos

While acceptable as a first attempt, the construction of Cob_d in the previous section has at least two major drawbacks. First, in order to define the horizontal composition of 1-bordisms (and hence 2-bordisms as well) we were required to use a vast application of the axiom of choice and choose collars arbitrarily for each 1-bordism. This could be avoided by using one of the so-called *non-algebraic* definitions of bicategory, [Lei02, Lei04, CL04], where providing specific compositions is not necessary. We will not do this. More importantly, we will want to consider variations on the bordism bicategory which involve placing additional structures on our bordisms, such as orientations, spin structures, or principal bundles, and these can be difficult to incorporate into our previous attempt in a satisfactory manner.

Both of these problems can be solved by modifying our bordisms (and objects) to include germs of neighborhoods of d-manifolds. We call these neighborhood germs *halos* and they easily generalize to include additional structures ($\mathcal{F}$-halos). The construction of the bordism bicategory then precedes in essentially the same manner as in the last section, using haloed $(d-2)$-manifolds, haloed 1-bordisms, and haloed 2-bordisms. Less care is needed in the gluing constructions, and we obtain symmetric monoidal bicategories Bord_d and $\mathsf{Bord}_d^{\mathcal{F}}$.

Definition 4.2.1. Let Y be a closed $(d-2)$-manifold or, respectively, a $(d-1)$-manifold with boundary. A *halation* for Y is a diagram of inclusions of manifolds,

$$Y \hookrightarrow A^{d-1} \hookrightarrow B^d,$$

together with orientations of the normal bundles $\nu(Y, A)$ and $\nu(A, B)|_Y$, where A is a $(d-1)$-manifold without corners or boundary (but possibly non-compact) and B is a d-manifold without corners or boundary. Halations for Y are partially ordered with partial order given by diagrams,

$$\begin{tikzcd}
 & A' \arrow[r, hook] \arrow[dd, hook] & B' \arrow[dd, hook] \\
Y \arrow[ur, hook] \arrow[dr, hook] & & \\
 & A \arrow[r, hook] & B
\end{tikzcd}$$

which induce orientation preserving isomorphisms of normal bundles $\nu(Y, A) \cong \nu(Y, A')$ and $\nu(A, B)|_Y \cong \nu(A', B')|_Y$. Similarly, a morphism of $(d-2)$-manifolds, or respectively $(d-1)$-manifolds, with halation is a commutative diagram of inclusions,

$$\begin{tikzcd}
Y \arrow[r, hook] \arrow[d, hook, "f"'] & A \arrow[r, hook] \arrow[d, hook, "f_A"'] & B \arrow[d, hook, "f_B"'] \\
Y' \arrow[r, hook] & A' \arrow[r, hook] & B'
\end{tikzcd}$$

which indices orientation preserving isomorphisms of bundles, $\nu(Y, A) \cong f^*\nu(Y', A')$ and $\nu(A, B)|_Y \cong f^*\nu(A', B')|_Y$. ◇

Halations can be visualized as in Figure 4.4. The central dot represents the $(d-2)$-manifold Y. The orientations of the normal bundles are distinguished by the different shadings in the neighborhood of Y. Halations for $(d-1)$-manifolds can similarly be visualized.

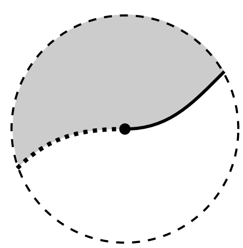


Figure 4.4: Halations

Definition 4.2.2. A *halo* around Y is an equivalence class of halations, Ψ, where the equivalence relation on halations for Y is generated by the partial order. Thus, two halations, $(Y \subset A \subset B)$ and $(Y \subset A' \subset B')$ are *equivalent* if there exists a third halation $(Y \subset A'' \subset B'')$ and a commutative diagram of inclusions,

$$
\begin{tikzcd}
 & A \arrow[r, hook] & B \\
Y \arrow[ur, hook] \arrow[r, hook] \arrow[dr, hook] & A'' \arrow[u, hook] \arrow[r, hook] \arrow[d, hook] & B'' \arrow[u, hook] \arrow[d, hook] \\
 & A' \arrow[r, hook] & B'
\end{tikzcd}
$$

inducing orientation preserving isomorphisms of normal bundles $\nu(Y, A) \cong \nu(Y, A')$ and $\nu(A, B)|_Y \cong \nu(A', B')|_Y$. A $(d-2)$- or $(d-1)$-manifold equipped with a halo is called a *haloed manifold*.

A morphism of haloed manifolds is an equivalence class of morphisms defined between halations which are representatives of the respective halos. Two morphisms of halations,

$$
\begin{aligned}
(f, f_A, f_B) &: (Y \subset A_Y \subset B_Y) \to (Z \subset A_Z \subset B_Z) \quad \text{and} \\
(f, f'_{A'}, f'_{B'}) &: (Y \subset A'_Y \subset B'_Y) \to (Z \subset A'_Z \subset B'_Z)
\end{aligned}
$$

are equivalent if there exists a third morphism of halations

$$(f, f''_{A''}, f''_{B''}) : (Y \subset A''_Y \subset B''_Y) \to (Z \subset A''_Z \subset B''_Z)$$

together with diagrams of inclusions,

$$
\begin{tikzcd}
 & A_Y \arrow[r, hook] & B_Y \\
Y \arrow[ur, hook] \arrow[r, hook] \arrow[dr, hook] & A''_Y \arrow[u, hook] \arrow[r, hook] \arrow[d, hook] & B''_Y \arrow[u, hook] \arrow[d, hook] \\
 & A'_Y \arrow[r, hook] & B'_Y
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A_Z \arrow[r, hook] & B_Z \\
Z \arrow[ur, hook] \arrow[r, hook] \arrow[dr, hook] & A''_Z \arrow[u, hook] \arrow[r, hook] \arrow[d, hook] & B''_Z \arrow[u, hook] \arrow[d, hook] \\
 & A'_Z \arrow[r, hook] & B'_Z
\end{tikzcd}
$$

such that the following additional diagrams commute,

$$
\begin{tikzcd}
A_Y \arrow[d, "f'_{A'}"] & A''_Y \arrow[l, hook'] \arrow[r, hook] \arrow[d, "f''_{A''}"] & A'_Y \arrow[d, "f_A"] \\
A_Z & A''_Z \arrow[l, hook'] \arrow[r, hook] & A'_Z
\end{tikzcd}
\qquad
\begin{tikzcd}
B_Y \arrow[d, "f'_{B'}"] & B''_Y \arrow[l, hook'] \arrow[r, hook] \arrow[d, "f''_{B''}"] & B'_Y \arrow[d, "f_B"] \\
B_Z & B''_Z \arrow[l, hook'] \arrow[r, hook] & B'_Z
\end{tikzcd}
$$

Haloed $(d-2)$-manifolds and $(d-1)$-manifolds form categories with the obvious notion of composition. ◇

Example 4.2.3. Given a $(d-2)$-manifold Y, there is a canonical halo given by the equivalence class of $Y \times \{(0,0)\} \subset Y \times \mathbb{R} \times \{0\} \subset Y \times \mathbb{R}^2$, with the standard orientations of the normal bundles. ◇

Definition 4.2.4. Let S^d be a d-manifold (possibly with corners). A *halation* for S is an inclusion of manifolds $S \hookrightarrow B^d$, where B is a d-manifold without boundary. A morphism of d-manifolds with halations is a diagram of inclusions,

$$\begin{tikzcd} S \arrow[r, hook] \arrow[d, hook, "f"'] & B \arrow[d, hook, "f_B"] \\ S' \arrow[r, hook] & B' \end{tikzcd}$$

and again halations on S are partially ordered with order given by those morphisms in which $f = id_S$. ◇

Definition 4.2.5. Let S^d be a d-manifold (possibly with corners). A *halo* for S is an equivalence class, Σ, of halations for S, where the equivalence relation on halations is generated by the partial order. A d-manifold equipped with a halo is called *haloed* and a morphism between haloed d-manifolds are given by equivalence classes of morphisms between representative halations. Two representative morphisms,

$$\begin{tikzcd} S \arrow[r, hook] \arrow[d, hook, "f"'] & B_S \arrow[d, hook, "f_B"] \\ T \arrow[r, hook] & B_T \end{tikzcd} \qquad \begin{tikzcd} S \arrow[r, hook] \arrow[d, hook, "f"'] & B'_S \arrow[d, hook, "f'_B"] \\ T \arrow[r, hook] & B'_T \end{tikzcd}$$

are equivalent if there exists a third morphism of halations $(f, f''_B) : (S, B''_S) \to (T, T''_V)$ such that the following diagram commutes:

$$\begin{tikzcd} B_S & B''_S \arrow[l, hook'] \arrow[r, hook] \arrow[d, "f''_B"] & B'_S \arrow[d, "f_B"] \\ B_T \arrow[from=1-1, "f'_B"] & B''_T \arrow[l, hook'] \arrow[r, hook] & B'_T \end{tikzcd}$$

Haloed d-manifolds form a category in the obvious way. ◇

Remark 4.2.6. Definition 4.2.5 makes sense with S^d, replaces by an $(d-1)$-manifold or a $(d-2)$-manifold as well. We will call such manifolds *partially haloed*. ◇

Let Ω be a haloed $(d-1)$-manifold and let Y be a closed component of the boundary. Then Y can be given the structure of a haloed $(d-2)$-manifold in two canonical ways. If $(W \subset A_W \subset B_W)$ is a representative halation for Ω, then we have a canonical diagram of inclusions

$$Y \hookrightarrow A_W \hookrightarrow B_W$$

and the normal bundle $\nu(A_W, B_W)|_Y$ inherits an orientation from the halo of W. To make this a halation for Y, we need only provide an orientation of the normal bundle $\nu(Y, A_W) \cong \nu(Y, W)$. There are two possible orientations corresponding to the *inward* pointing normal vector and the *outward* pointing normal vector and this gives us two haloed $(d-2)$ mandifolds Ψ_{in} and Ψ_{out}.

If $\Omega = [W \subset A_W \subset B_W]$ is a haloed $(d-1)$-manifold with a decomposition of the boundary $\partial W = \partial_{\text{in}} W \sqcup \partial_{\text{out}} W$, of its underlying manifold, W, then we make the convention that $\partial_{\text{in}}\Omega$ is the haloed $(d-2)$-manifold given by $\partial_{\text{in}} W$ with its inward halo. Similarly we define $\partial_{\text{out}}\Omega$ as $\partial_{\text{out}} W$ with its outward halo. With this convention it is easy to define haloed 1-bordisms.

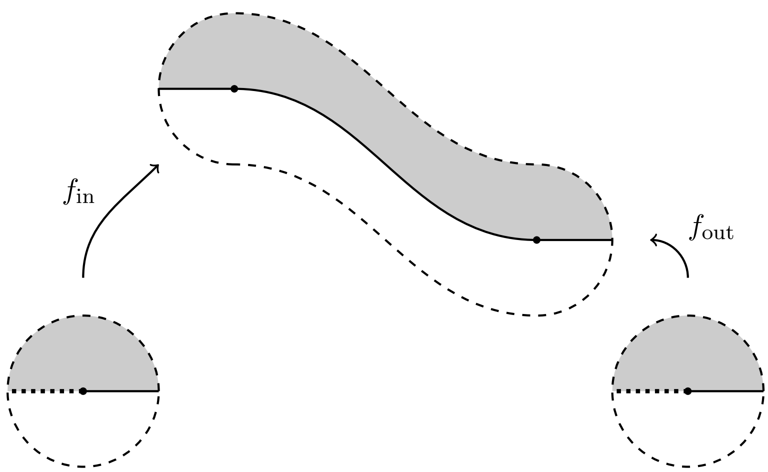


Figure 4.5: A Haloed 1-Bordisms

Definition 4.2.7. A *haloed 0-bordism* is a $(d-2)$-manifold Y equipped with the canonical halo $\Psi = [Y \subset Y \times \mathbb{R} \subset Y \times \mathbb{R}^2]$. Let Ψ_0 and Ψ_1 be haloed 0-bordisms. A *haloed 1-bordism* from Ψ_0 to Ψ_1 consists of a haloed $(d-1)$-manifold Ω, with a decomposition of its boundary

$\partial\Omega = \partial_{\mathrm{in}}\Omega \sqcup \partial_{\mathrm{out}}\Omega$ (with the conventions as above), together with isomorphisms of haloed $(d-2)$-manifolds, $\Psi_0 \cong \partial_{\mathrm{in}}\Omega$ and $\Psi_1 \cong \partial_{\mathrm{out}}\Omega$. ◇

Example 4.2.8. Let $\Psi = [Y \subset Y \times \mathbb{R} \times \{0\} \subset \mathbb{R}]$ be a 0-bordism. Then define the halation $\Psi \times I$ by

$$Y \times I \times \{0\} \hookrightarrow Y \times \mathbb{R} \times \{0\} \hookrightarrow Y \times \mathbb{R}^2.$$

The haloed $(d-1)$-manifold $\Psi \times I$, together with the obvious inclusions of Ψ at both boundaries, defines a haloed 1-bordism from Ψ to itself.

We will also define the following variation. Let $\Psi \tilde{\times} I$ denote the class of the halation,

$$Y \times \{0\} \times I \hookrightarrow Y \times \{0\} \times \mathbb{R} \hookrightarrow Y \times \mathbb{R}^2.$$

This is isomorphic to the previous haloed manifold, however there the natural inclusions of Ψ do *not* make this a haloed 1-bordism; the $(d-1)$-manifolds are at right angles, and hence incompatible. However it does give a *partially haloed* 1-bordism, as per Remark 4.2.6. ◇

We wish to also define haloed 2-bordisms, however there is a problem. The natural thing to try is to take a haloed $\langle 2\rangle$-manifold $\Sigma = [S \subset B^d]$ with a decompositions of its boundaries, $\partial_0 S = \partial_{0,\mathrm{in}} S \sqcup \partial_{0,\mathrm{out}} S$ and $\partial_1 S = \partial_{1,\mathrm{in}} S \sqcup \partial_{1,\mathrm{out}} S$. One would then like to define a haloed 2-bordism using isomorphisms of haloed $(d-1)$-manifolds,

$$\begin{aligned} \partial_{0,\mathrm{in}} S \sqcup \partial_{0,\mathrm{out}} S &\cong \Omega_0 \sqcup \Omega_1 \\ \partial_{1,\mathrm{in}} S \sqcup \partial_{1,\mathrm{out}} S &\cong \Psi_0 \times I \sqcup \Psi_1 \times I. \end{aligned}$$

Herein lies the problem. While the boundaries $\partial_0 S$ and $\partial_1 S$ have a canonical germs of d-manifolds around them, they don not have canonical germs of $(d-1)$-manifolds (without boundary).[1] Thus we must be more careful in constructing haloed 2-bordisms.

Definition 4.2.9. Let $\Sigma = [S \subset B^d_S]$ be a haloed d-dimensional $\langle 2\rangle$-manifold, and $\Omega = [W \subset A^{d-1}_W \subset B^d_W]$ be a haloed $(d-1)$-manifold. Then a *map* from Ω to $\partial_0\Sigma$ is defined to be an equivalence class of commutative diagrams,

[1]For example, consider the haloed $\langle 2\rangle$-manifold consisting of the second quadrant sitting in the plane. Consider the portion of the boundary, X, consisting of the negative x-axis. This has a canonical germ of a 2-manifold, given by the embedding in the plane. But we can extend X to a 1-manifold without boundary in multiple ways. For example, we can extend by embedding it in the x-axis, or we can extend by the graph of the smooth function,

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

These yield distinct halations for X. They are isomorphic, but not canonically so.

$$\begin{tikzcd}
W \arrow[r, hook] \arrow[d, "g"'] & A_W \arrow[r, hook] & B_W \arrow[d, "g_B"] \\
\partial_0 S \arrow[r, hook] & S \arrow[r, hook] & B_S
\end{tikzcd}$$

where g is an isomorphism of W with a component of $\partial_0 S$, and g_B is an embedding. Two such diagrams (g, g_B) and (g', g'_B) are equivalent if $g = g'$ identically and $g'_B = g_B$ when restricted to some possibly smaller neighborhood B'.

A map $(g, g_B) : \Omega \to \partial_0 \Sigma$ is called *inward* if the orientation of $\nu(W, B_W)$ agrees with the inward pointing orientation of $\nu(\partial_0 S, B_S)$, under the isomorphism induced by (g, g_B). If the orientation disagrees, we say (g, g_B) is *outward*. Inward and outward maps are defined for partially haloed manifolds in the analogous fashion. $\diamond$

Definition 4.2.10. Let Ψ_0 and Ψ_1 be two 0-bordisms and let Ω_0 and Ω_1 be two 1-bordisms from Ψ_0 to Ψ_1. A *haloed 2-bordism* from Ω_0 to Ω_1 consists of:

1. A haloed d-dimensional $\langle 2 \rangle$-manifold, $\Sigma = [S \subset B^d_S]$,

2. Decompositions:

$$\partial_{0,\text{in}} S \sqcup \partial_{0,\text{out}} S,$$
$$\partial_{1,\text{in}} S \sqcup \partial_{1,\text{out}} S.$$

3. An inward map,

$$(f_0, f_{0,B}) : \Psi_0 \tilde{\times} I \to \partial_{1,\text{in}} \Sigma,$$

   and an outward map,

$$(f_1, f_{1,B}) : \Psi_1 \tilde{\times} I \to \partial_{1,\text{in}} \Sigma,$$

   where $\Psi_i \tilde{\times} I$ denotes the partially haloed $(d-1)$-manifold of Example 4.2.8.

4. An inward map,

$$(g_0, g_{0,B}) : \Omega_0 \to \partial_{0,\text{in}} \Sigma,$$

   and an outward map,

$$(g_1, g_{1,B}) : \Omega_1 \to \partial_{0,\text{in}} \Sigma.$$

such that $(S, f_0 \sqcup f_1, g_0 \sqcup g_1)$ is a 2-bordism in the sense of Definition 4.1.11 and all the obvious diagrams commute. An *isomorphism* of haloed 2-bordisms is a morphism $\Sigma \to \Sigma'$ of haloed $\langle 2 \rangle$-manifolds making all the relevant diagrams commute. $\diamond$

Remark 4.2.11. There is a completely analogous notion of pseudo-isotopy for maps of haloed manifolds and Lemma 4.1.13 carries through just as before to show that the isomorphism class of a holoed 2-bordisms only depends on the pseudo-isotopy class of $f = f_0 \sqcup f_1$. $\diamond$

Gluing haloed bordisms is much easier then for ordinary bordisms because unlike the usual category of manifolds the relevant push outs *do exist* in the category of haloed manifolds.

Proposition 4.2.12. *Let Ω_a be a haloed 1-bordism from Ψ_0 to Ψ_1 and Ω_b be a haloed 1-bordism from Ψ_1 to Ψ_2. Then the following push out exists,*

$$\begin{tikzcd} \Psi_1 \arrow[r] \arrow[d] & \Omega_a \arrow[d] \\ \Omega_b \arrow[r] & \Omega_b \circ \Omega_a \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

and with the natural inclusions of Ψ_0 and Ψ_2, forms a 1-bordism from Ψ_0 to Ψ_2.

Proof. The pushout can be computed by choosing representatives for the haloed manifolds. Without loss of generality we may assume these representatives are of the form $M \times \mathbb{R}$ for some $(d-1)$-manifold M. Thus it is enough to restrict our attention to just this $(d-1)$-dimensional portion. The geometry of the haloed 0-bordism implies the $(d-1)$-portion of Ψ_1 splits into two pieces $\Psi_1 = \Psi_+ \cup_Y \Psi_-$, where Y is the underlying $(d-2)$-manifold of Ψ_1. Moreover the above pushout diagram becomes (focusing on just the $(d-1)$-dimensional aspect)

$$\begin{tikzcd} \Psi_+ \cup_Y \Psi_- \arrow[r, hook] \arrow[d, hook] & \Psi_+ \cup_Y M_a \arrow[d, hook] \\ M_b \cup_Y \Psi_- \arrow[r, hook] & \Omega_b \circ \Omega_a \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

An easy exercise shows that such a puchout is always the same as the pushout,

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d, hook] & M_b \arrow[d, hook] \\ M_a \arrow[r, hook] & M_b \cup_Y M_a \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

Thus the $(d-1)$-dimensional portion (and hence all) of $\Omega_b \circ \Omega_a$ is a Hausdorff topological manifold.

The commutativity of the d-dimensional square implies that there is only one smooth structure which is compatible with the map from the haloed manifolds. Moreover the haloes of the manifolds Ψ_1, Ω_a, and Ω_b induce a halo for $\Omega_b \circ \Omega_a$. It is easy to verify that this haloed manifold satisfies the universal property. The last statement is equally clear. □

Vertical and horizontal composition of haloed 2-bordisms can be similarly defined via push outs. This defines a bicategory Bord_d whose objects are haloed 0-bordisms, whose 1-morphisms are haloed 1-bordisms and whose 2-morphisms are isomorphism classes of haloed 2-bordisms. As before, the symmetric monoidal structures on the categories of haloed manifolds (given by disjoint union) induce the structure of a symmetric monoidal bicategory on Bord_d. Forgetting the halo structure gives a forgetful symmetric monoidal homomorphism $\mathsf{Bord}_d \to \mathsf{Cob}_d$.

Theorem 4.2.13. *The symmetric monoidal homomorphism* $\mathsf{Bord}_d \to \mathsf{Cob}_d$ *is an equivalence of symmetric monoidal bicategories.*

Proof. By Theorem 3.4.10, we only need to check that this homomorphism is essentially surjective on objects, essentially full on 1-morphisms, and fully-faithful on 2-morphisms. The first two of these are clear since it is actually surjective on objects and 1-morphisms. It is also full on 2-morphisms. It is faithful on 2-morphisms because to haloed 2-bordisms are isomorphic if and only if their underlying non-haloed 2-bordisms are isomorphic. □

We can now introduce structures into our bordism bicategory. Structures will only live on the d-dimensional part of a haloed manifold. Since composition in the bordism bicategory is give by gluing manifolds, what ever structure we wish to consider must also have a gluing property. In fact the category of d-manifolds with structure is required to be a stack over the category of d-manifolds. Since we want to construct a symmetric monoidal bicategory of bordisms with structure, these structures must behave well with respect to disjoint union.

Let Man^d be the category of d-manifolds without boundary, with embeddings as morphism. Man^d is a Grothendieck site with covering families the usual notion of covering, see Section C.1 in the appendix. Let $\mathcal{F}$ be a symmetric monoidal stack over Man^d, in the

sense of Appendix C. An object of $\mathcal{F}$ will be called an $\mathcal{F}$-manifold. We will think of an $\mathcal{F}$-manifold as a pair $(M, s \in \mathcal{F}(M))$, i.e. as a manifold M together with an $\mathcal{F}$-structure, s, on M. Replacing every occurrence of "d-manifold without boundary" in the definitions of haloed manifolds with "$\mathcal{F}$-manifold" one obtains the notion of an *$\mathcal{F}$-haloed manifold*. The definitions of haloed 0-bordisms and 1-bordisms go through verbatim for $\mathcal{F}$-haloed manifolds, as well.

The composition of $\mathcal{F}$-1-bordisms can be defined, just as before, via push out. The fact that $\mathcal{F}$ is a stack ensures that we will always be able to equip the glued bordism with an $\mathcal{F}$-structure and that this $\mathcal{F}$-structure is unique up to unique isomorphism.

The definition of haloed 2-bordism is problematic. The first main difficulty is that in the definition itself, for each of the object 0-bordisms, Ψ_0 and Ψ_1, we need to make sense of $\Psi_0 \times I$. While Example 4.2.8 shows this is possible for haloed manifolds, it is not clear how to proceed for general $\mathcal{F}$-manifolds. Even if this can be surmounted, as is the case in many examples, a related problem comes up when we try to define the vertical composition of 2-bordisms. We are able to glue the 2-bordisms without difficulty, but to turn the result into a new 2-bordism, we needed to choose and identification $I \cup_{pt} I \cong I$. In the case of haloed 2-bordisms, the result was independent of which choice of identification we made. In the presence of many examples of $\mathcal{F}$-structures this will not be the case.

One solution to this problem is to replace the bordism bicategory with a bordism *double category*. This was the approach used in [Mor07], and it solve both of the above problems. In this situation there are two kinds of 1-morphisms, vertical and horizontal, and a 2-morphism sits inside a square of 1-morphism. It is straightforward to use $\mathcal{F}$-haloed manifolds in this context. In this approach, one must also deal with the symmetric monoidal structure. We will not pursue this here. Instead we restrict ourselves to dealing with those $\mathcal{F}$-structures where the above problems are in fact not problems at all.

A topological $\mathcal{F}$-structure (defined below) will be a symmetric monoidal stack, $\mathcal{F}$, over Man^d equipped with some additional structure. This structure will be required to satisfy certain properties which allow us to mimic the construction of the haloed bordism bicategory, essentially without any significant change.

The first pice of structure is that we need a pair of functors

$$\times I : \{\mathcal{F}\text{-0-bordisms}\} \to \{\text{Partially Haloed } \mathcal{F}\text{-1-bordisms}\},$$
$$\times I : \{\mathcal{F}\text{-0-bordisms}\} \to \{\mathcal{F}\text{-1-bordisms}\},$$

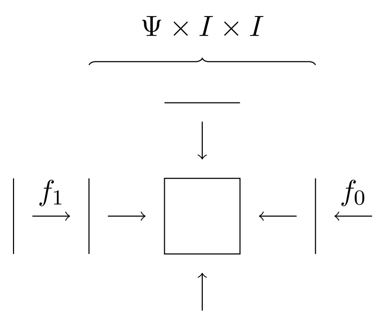
which lift the usual functors from unstructured 0-bordisms to unstructured 1-bordisms. Given this structure we can now *define* $\mathcal{F}$-2-bordisms, exactly as we did in Definition 4.2.10, replacing every occurrence of "haloed d-manifold" with "$\mathcal{F}$-haloed d-manifold". Morphisms of $\mathcal{F}$-2-bordisms are defined in the obvious way, and there is a functorial horizontal composition given by gluing $\mathcal{F}$-2-bordisms. Just as in the haloed case, there are canonical natural isomorphisms realizing associativity of this composition.

The second piece of structure that we will need is a functor,

$$\times I : \{\mathcal{F}\text{-1-bordisms}\} \to \{\mathcal{F}\text{-2-bordisms}\},$$

which lifts the usual functor between unstructured bordisms. This allows us to define $\mathcal{F}$-pseudo isotopy for $\mathcal{F}$-1-bordisms. $\mathcal{F}$-pseudo isotopy can be defined in general, but we will only need it in a vary particular case.

Definition 4.2.14. Let Ψ be an $\mathcal{F}$-0-bordism, and consider two automorphisms of the partially haloed $\mathcal{F}$-1-bordism $\Psi \times I$, f_0 and f_1. In particular f and g are the identity near the ends of $\Psi \times I$. This determines an $\mathcal{F}$-2-bordism, $\Psi \times I \times I$. We may modify this $\mathcal{F}$-2-bordism by pre-composition by f_0 and f_1, as in the diagram:



We say that f_0 and f_1 are *$\mathcal{F}$-pseudo isotopic* if there exists an isomorphism between this new $\mathcal{F}$-2-bordism and the original $\mathcal{F}$-2-bordism $\Psi \times I \times I$. ◇

Definition 4.2.15. A symmetric monoidal stack, $\mathcal{F}$, over Man^d will be called *topological* if it is equipped with functors,

$$\times I : \{\mathcal{F}\text{-0-bordisms}\} \to \{\text{Partially Haloed } \mathcal{F}\text{-1-bordisms}\}$$
$$\times I : \{\mathcal{F}\text{-0-bordisms}\} \to \{\mathcal{F}\text{-1-bordisms}\}$$
$$\times I : \{\mathcal{F}\text{-1-bordisms}\} \to \{\mathcal{F}\text{-2-bordisms}\}$$

lifting the usual $\times I$ functors, such that

1. For any two automorphisms, f_0 and f_1 of $\Psi \times Y$, as in Definition 4.2.14, if the underlying morphisms of unstructured bordisms are pseudo-isotopic, then f_0 and f_1 are $\mathcal{F}$-pseudo-isotopic.

2. If Σ is an $\mathcal{F}$-2-boridsm with source and target $\mathcal{F}$-0-bordisms Ψ_0 and Ψ_1, then the $\mathcal{F}$-2-bordism $(\Psi_1 \times I \times I) \circ \Sigma \circ (\Psi_0 \times I \times I)$ is isomorphic to Σ.

$\diamond$

The properties defining a topological $\mathcal{F}$-structure allow us to define an associative vertical composition and to provide associators and unitors for the horizontal composition of $\mathcal{F}$-1-bordism. These structure and the proof of associativity are exactly as for unstructured haloed manifolds, and so we omit the details of the proof. This yields a bicategory $\mathsf{Bord}_d^{\mathcal{F}}$ whose objects are $\mathcal{F}$-0-bordisms, whose 1-morphisms are $\mathcal{F}$-1-bordisms, and whose 2-morphisms are $\mathcal{F}$-2-morphisms. Since $\mathcal{F}$ was assumed to be a symmetric monoidal stack, we again get an induced structure realizing $\mathsf{Bord}_d^{\mathcal{F}}$ as a symmetric monoidal bicategory. There are many examples of topological $\mathcal{F}$-structures: Orientations, Spin Structures, G-principal bundles, etc. The main techniques used in proving the classification of unoriented 2-dimensional topological field theories carry over to the setting of topological $\mathcal{F}$-structures. After proving the unoriented classification, we demonstrate this by adapting our techniques to the oriented case, as well.

Remark 4.2.16. The unoriented topological field theories play an especially important role among all structured topological field theories. The are universal in the sense that they give rise to a field theory with *any* topological structure $\mathcal{F}$. More precisely the unoriented structure corresponds to $\mathcal{F} = \mathsf{Man}_d$, and thus is *terminal* in the bicategory of topological structures. In particular, for any topological structure $\mathcal{F}$ there is the forgetful symmetric monoidal homomorphism $\mathsf{Bord}_d^{\mathcal{F}} \to \mathsf{Bord}_d$. Thus, for any target symmetric monoidal bicategory C, there is a natural homomorphism

$$\mathsf{SymBicat}(\mathsf{Bord}_d, \mathsf{C}) \to \mathsf{SymBicat}(\mathsf{Bord}_d^{\mathcal{F}}, \mathsf{C}).$$

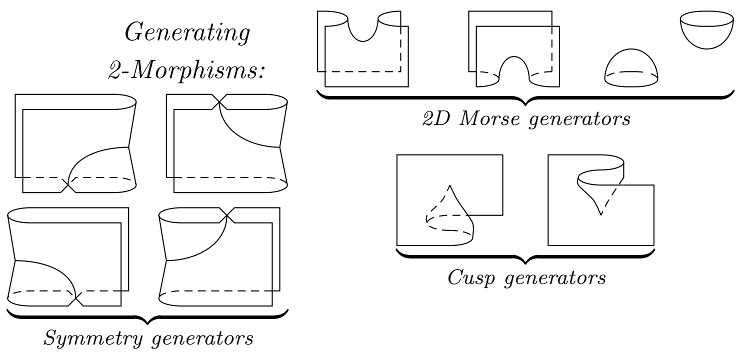
This expresses the universality of unoriented theories. $\diamond$

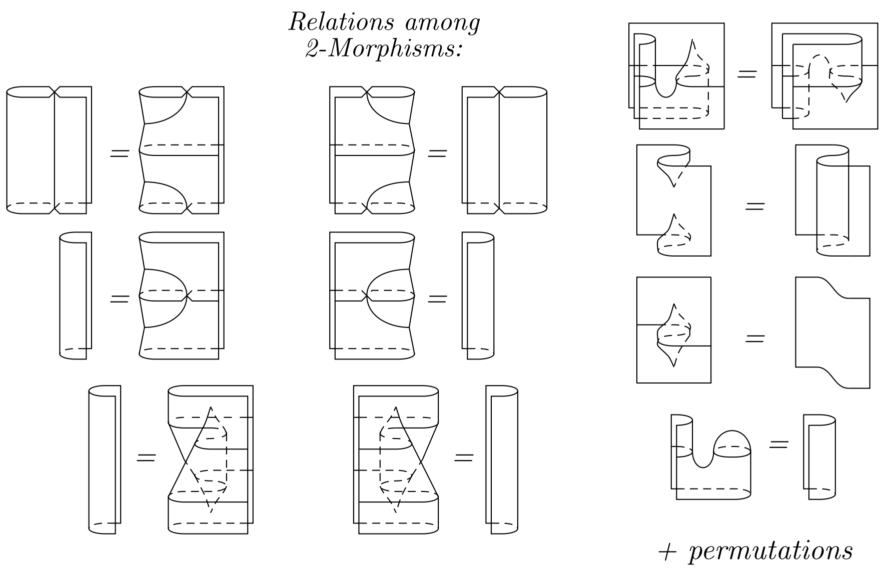
4.3 The Unoriented Classification

Theorem 4.3.1 (Classification of Unoriented Topological Field Theories)**.** *The unoriented bordism bicategory* Bord *has the following generators and relations presentation as a symmetric monoidal bicategory:*

Generating Objects: $\bullet$

Generating 1-Morphisms: $\supset$ $\subset$





Proof. Definition 3.5.4 gives a precise meaning to the abstract symmetric monoidal bicategory $\mathsf{F}/\mathcal{R}$ with the above generators and relations and Propositions 3.5.7 and 3.6.1 provide an easy mechanism for constructing symmetric monoidal homomorphisms out of this bicategory; we need only specify the value of the generators in the target bicategory.

We have drawn the abstract generators of this bicategory as particular bordisms, and by the above considerations this defines a symmetric monoidal homomorphism E :

$\mathsf{F}/\mathcal{R} \to \mathsf{Bord}$. The objects, 1-morphisms and 2-morphisms of $\mathsf{F}/\mathcal{R}$ are given by binary words, binary sentences, and paragraphs in the abstract generating data (modulo the relations) and the homomorphism E is given by evaluating the given word, sentence, or paragraph in Bord. We must show that this is an equivalence of symmetric monoidal bicategories.

Whitehead's Theorem for symmetric monoidal bicategories (Theorem 3.4.10) provides a simple list of criteria for when such a symmetric monoidal homomorphism is an equivalence. We must show three things: (1) E must be essentially surjective on objects, (2) E must be essentially full on 1-morphisms, (3) E must be fully-faithful on 2-morphisms.

The first of these is the statement that every (compact) 0-manifold is equivalent in Bord to a finite disjoint union of points. This is clear. The second statement is the assertion that any 1-bordism is isomorphic in Bord to one in the image of E, i.e. one obtained by gluing elementary 1-bordisms. The existence of Morse functions ensures this is the case. Finally we must show that E is fully-faithful on 2-morphisms. This is the content of the Planar decomposition Theorem 2.5.7. □

4.4 Additional Structure: The Oriented Classification

In this section we show how to adapt the previous results to classify field theories for bordisms with structure. We focus on the oriented bordism category as a concrete example. Let us review the main steps in the proof of Theorem 4.3.1, and see how they must be adapted in the setting of $\mathcal{F}$-structures.

The generators and relations theorems for symmetric monoidal bicategories allow us to construct abstract symmetric monoidal bicategories which are easy to map out of. Thus it is relatively easy to construct an abstract candidate symmetric monoidal bicategory $\mathsf{B}^{\mathcal{F}}$ and a symmetric monoidal homomorphism $h : \mathsf{B}^{\mathcal{F}} \to \mathsf{Bord}_2^{\mathcal{F}}$. The main difficulty is to show that this is an equivalence. For this we must appeal to Theorem 3.4.10, which gives three criteria characterizing symmetric monoidal equivalences. We must have that h to be essentially surjective on objects, essentially full on 1-morphisms, and fully faithful on 2-morphisms.

First we need h to be essentially surjective on objects. Since any 0-dimensional $\mathcal{F}$-manifold is a disjoint union of haloed points with $\mathcal{F}$-structure, it is enough to identify these. Moreover we only need to understand them up to isomorphism *in the bordism bicategory.* In other words, we want to classify $\mathcal{F}$-structures on the point up to invertible $\mathcal{F}$-bordism.

Step 1. *Identify the $\mathcal{F}$-structures on the point, up to invertible $\mathcal{F}$-bordism.*

After we do this, we will pick a representative for each equivalence class and these will be the generating objects of $\mathsf{B}^{\mathcal{F}}$. $h : \mathsf{B}^{\mathcal{F}} \to \mathsf{Bord}_2^{\mathcal{F}}$ will then necessarily be essentially surjective on objects. In order to accomplish this first step, we must also have some rudimentary understanding of the $\mathcal{F}$-bordisms, themselves. This is important anyway, since the second criteria for h to be an equivalence is that it is essentially full on 1-morphisms. Just as we understood the $\mathcal{F}$-structure on the points, we will need to understand the $\mathcal{F}$-structures on the 1-bordisms.

Every 1-bordism is isomorphic to a disjoint union of intervals and left and right "elbows". Thus it is enough to understand the $\mathcal{F}$-structures on these. Again, we only need to understand them up to isomorphism in the bordism bicategory. Choosing representatives will lead to a collection of generating 1-morphisms for $\mathsf{B}^{\mathcal{F}}$. The homomorphisms h will now be essentially full on 1-morphisms.

Step 2. *Identify the $\mathcal{F}$-structures on the interval, and on the left and right elbows, up to $\mathcal{F}$-bordism.*

Finally, we will need to prove that $h : \mathsf{B}^{\mathcal{F}} \to \mathsf{Bord}_2^{\mathcal{F}}$ is fully-faithful on 2-morphisms. In the unoriented case, we proved this using the Planar Decomposition Theorems 2.5.7 and 2.5.6. To prove this in the setting of $\mathcal{F}$-structures we must adapt the planar decomposition theorem to the setting of $\mathcal{F}$-structures. The Planar Decomposition Theorem decomposes into simple parts. First there is the local analysis of maps to $\mathbb{R}^2$. This doesn't change in the setting of $\mathcal{F}$-structures.

The local analysis allows us to do several things. Given an $\mathcal{F}$-2-bordism, we may choose a generic map to $\mathbb{R}^2$. Via the graphic, and after making some choices, this gives us a way to cup up our surface, in a bicategorical fashion, into elementary pieces. We must first understand what $\mathcal{F}$-structures are allowed on these pieces.

Step 3. *Identify the isomorphism classes (rel. boundary) of $\mathcal{F}$-structures on the elementary 2-bordisms.*

Secondly, we must understand how to glue these elementary pieces back together to recover the original $\mathcal{F}$-2-bordism. In the unoriented case, this gluing information was encoded in what we called *sheet data*. There will be an analogous Planar Decomposition

Theorem in the setting of $\mathcal{F}$-structures and the main difference between this and the unoriented Planar Decomposition Theorem is in the sheet and gluing data.

In fact, we have already started the process of determining the new sheet data in the presence of $\mathcal{F}$-structures. Consider a small ball in the graphic which contains no critical points. The inverse image of this region consists of a disjoint union of disks with $\mathcal{F}$-structure. Choosing a point p, in each disk we get a $\mathcal{F}$-structure on a neighborhood of this point, which in turn gives us an $\mathcal{F}$-0-bordism. Each disks is isomorphic to $p \times I \times I$. Thus the new sheet data that we should assign to a region without critical points as a collection of sets, one for each of the generating objects we identified in Step 1. Similarly Steps 2 and 3 tell us precisely what sheet data we should associate along folds, along fold crossings, along cusps, and along 2D Morse critical points.

The next thing to understand is the appropriate gluing data that we should assign to overlaps. But this is also part of Step 2, namely the part which understands the $\mathcal{F}$-structures on intervals. Similarly there will be cocycle conditions on the triple intersections.

Step 4. *Use the previous steps to determine the sheet and gluing data for planar decompositions of surfaces with $\mathcal{F}$-structure.*

Finally, we must understand when two of these $\mathcal{F}$-planar decompositions result in the same $\mathcal{F}$-manifold. Theorem 2.5.6 gives a precise list of local moves which allow us to pass from one planar decomposition to another. We simply need to understand how to add in appropriate sheet data.

Step 5. *Determine which $\mathcal{F}$-sheet data is appropriate for the elementary moves of Theorem 2.5.6.*

After completing these steps one obtains a new version of the planar decomposition theorem. The sheet and gluing data is augmented with additional $\mathcal{F}$-structure information. The generating 2-morphisms and relations of $\mathsf{B}^{\mathcal{F}}$ are now determined by this sheet data, and consequently $h : \mathsf{B}^{\mathcal{F}} \to \mathsf{Bord}_2^{\mathcal{F}}$ will be fully faithful on 2-morphisms, and hence a symmetric monoidal equivalence.

To demonstrate that this program is actually feasible, the rest of this section is devoted to carrying it out in the case of orientations. Let Or_d be the symmetric monoidal category of oriented d-manifolds, with embeddings as morphisms. The forgetful functor $\mathrm{Or}_d \to \mathsf{Man}^d$, realizes this as a symmetric monoidal stack over Man^d which is topological in the sense of Definition 4.2.15. In fact it is a sheaf; orientations have no automorphisms.

The five steps outlined in this section are made easier in the oriented case by the following observation. A 2-manifold which is topologically a disk has exactly two orientations. This implies that there are exactly two oriented 0-bordisms whose underlying zero manifold is a point. Let us fix two representatives, call them the *positive point* and the *negative point*. Similarly the haloed interval, viewed as a haloed 1-bordism from the haloed point to itself supports precisely two orientations. These correspond to the identity 1-bordisms from the positive point, respectively negative point, to itself. In particular there are no oriented 1-bordisms from the positive point to the negative point. Thus we see that even up to oriented bordism there are precisely two orientations on the point, see Figure 4.6.

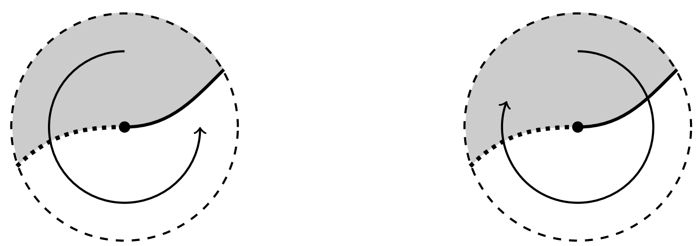


Figure 4.6: Two Oriented 0-Bordisms

To complete the second step, we must understand the orientations on the right and left "elbows". Let us start with the elbow whose source consists of the disjoint union of two points and whose target is the empty set. There are four possible orientations on the disjoint union of two points, and only two of these are compatible with orientations of the elbow. See Figure 4.7 for an example. The sources of these two oriented elbows are either $pt^+ \sqcup pt^-$ or $pt^- \sqcup pt^+$. Thus there are a priori four generating 1-bordisms. We will see later, however, that two of these generators are redundant and that we can eliminate them and get an equivalent symmetric monoidal bicategory.

Each of the elementary 2-bordisms is topologically a disk, so again each has exactly two orientations. Given an orientation of the boundary of an elementary 2-bordism, there is precisely one orientation on the 2-bordism extending this orientation. Thus the elementary 2-bordisms are just as they are for the unoriented bordism bicategory, except that the boundaries are equipped with orientations. This completes Step 3.

We can now describe how to modify the notion of a planar diagram in order to

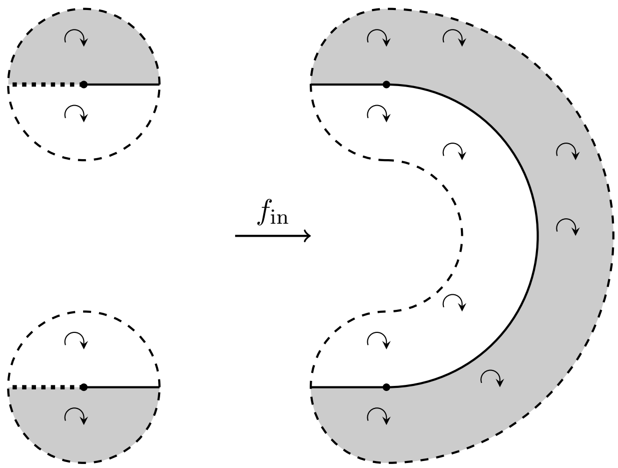


Figure 4.7: An Oriented 1-Bordism

incorporate orientations. The main difference between the oriented and unoriented cases will be in the sheet data. The oriented sheet data for regions without critical points now consists of *two* sets of sheets, those corresponding to the positively oriented point and those corresponding to the negatively oriented point. Similarly the sheet data for the fold and 2D Morse regions consist of a quadruple of sets $(S_+, S_-, \{t_+\}, \{t_-\})$ such that $(S_+ \sqcup S_-, \{t_+, t_-\})$ forms sheet data in the unoriented sense. The cusp crossing sheet data is similar. Finally cusp sheet data consists of a pair of maps of pairs of sets

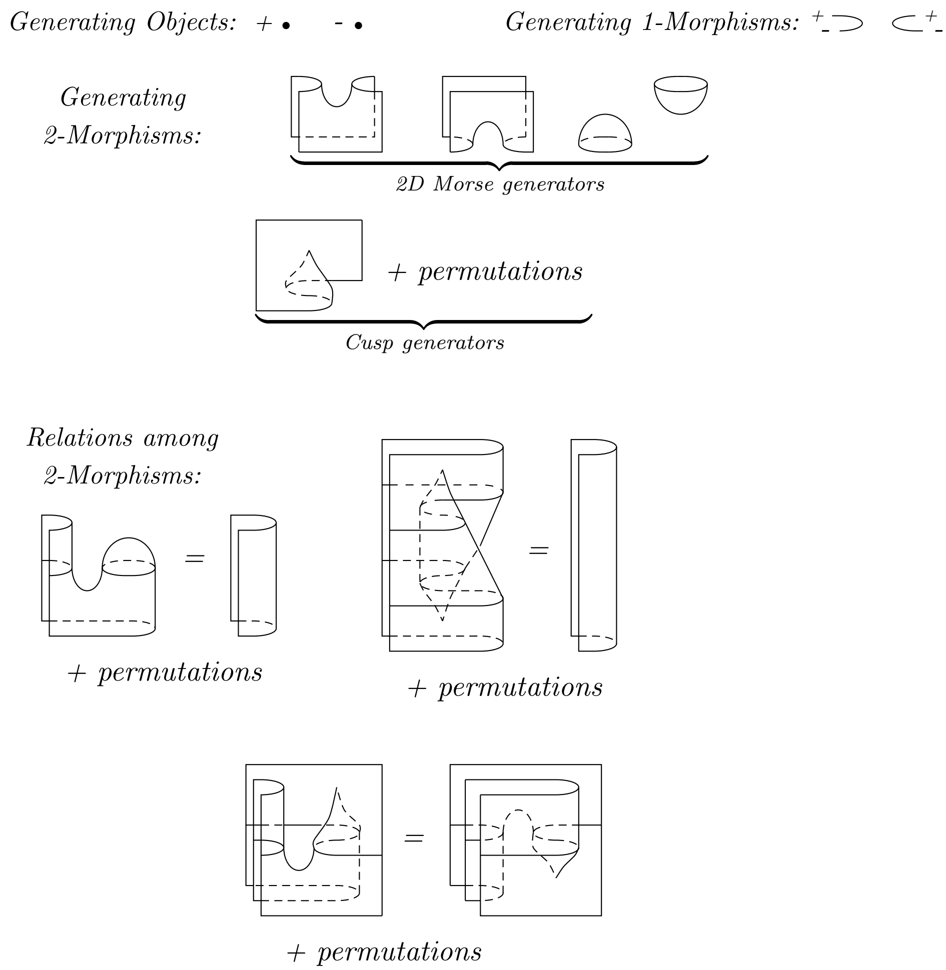
$$(S_+, S_-) \rightrightarrows (S'_+, S'_-).$$

These are required to satisfy that $(S_+ \sqcup S_-) \rightrightarrows (S'_+ \sqcup S'_-)$ consists of unoriented cusp sheet data.

The gluing data for regions without singularities is given simply by a bijection of pairs of sets, $(S_+, S_-) \cong (\overline{S}_+, \overline{S}_-)$. On triple intersections these must satisfy the obvious cocycle condition. The gluing data for regions which contain fold singularities are more interesting. The gluing data from $(S_+, S_-, \{t_+\}, \{t_-\})$ to $(\overline{S}_+, \overline{S}_-, \{\overline{t}_+\}, \overline{\{t_-\}})$, consists of a pair of bijections $S_+ \cong \overline{S}_+$, $S_- \cong \overline{S}_-$. Not automorphisms of $T = \{t_+, t_-\}$ are allowed. Translating this into generators for the oriented bordism bicategory, we see that this has the effect of eliminating the symmetry generators from the list of generating 2-morphisms. A similar analysis shows that each of the relations for the unoriented bordism bicategory holds whenever the oriented bordisms are composable. Putting these results together yields

the following oriented classification theorem.

Theorem 4.4.1. *The oriented 2-dimensional bordism bicategory Bord_2^{or} has the following generators and relations as a symmetric monoidal bicategory:*



4.5 Transformations and Modifications of TFTs

Now that we have deduced a generators and relations presentation of the oriented and unoriented bordism bicategories can understand very concretely the bicategories of homomorphisms out of these categories, i.e. the bicategories of oriented and unoriented TFTs. Propositions 3.5.7 and 3.6.1 are essential for this purpose. They allow us to characterize this homomorphism bicategories in terms of a specific small about of data which corresponds to the images of the generators in the target symmetric monoidal bicategory. In this section

we will collect together a few basic results and transformations and modifications between topological field theories, focusing only on the oriented case. The existence of the forgetful symmetric monoidal homomorphism $\mathsf{Bord}_2^{\mathrm{or}} \to \mathsf{Bord}_2$ implies that any unoriented topological field theory gives rise to an oriented theory, so the results of this section are also valid in the unoritented setting.

Let us fix a target symmetric monoidal bicategory C, and assume that we have two topological field theories Z_0 and Z_1 with values in C. We will first work in the oriented case, and then explain what happens in the unoriented setting. Propositions 3.5.7 and 3.6.1 allow us to characterize the transformations between Z_0 and Z_1. They are determined by their values on the generating objects and 1-morphisms. In the oriented case there are only two generating objects and two generating 1-morphisms. Thus a transformation between TFTs is given by specifying data:

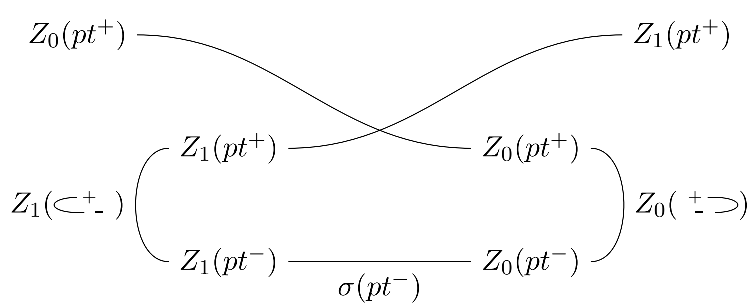
$$\begin{aligned}
\sigma(pt^+) &: Z_0(pt^+) \to Z_1(pt^+) \\
\sigma(pt^-) &: Z_0(pt^-) \to Z_1(pt^-) \\
\sigma(\subset^{+}_{-}) &: Z_1(\subset^{+}_{-}) \circ \sigma(pt^+ \sqcup pt^-) \to \sigma(\emptyset) \circ Z_0(\subset^{+}_{-}) \\
\sigma({}^{+}_{-}\supset) &: Z_1({}^{+}_{-}\supset) \circ \sigma(\emptyset) \to \sigma(pt^+ \sqcup pt^-) \circ Z_0({}^{+}_{-}\supset)
\end{aligned}$$

This data is subject to a number of relations, each coming from one of the generating 2-morphisms.

The significance of this data will become clearer if we repackage it slightly. We can express it entirely in terms of data involving only the images of the positively oriented point, pt^+. The 1-morphism $\sigma(pt^-) : Z_0(pt^-) \to Z_1(pt^-)$ in C gives rise to a new 1-morphism from $Z_1(pt^+)$ to $Z_0(pt^+)$, via the composition,

$$\tilde{\sigma}(pt^+) := [Z_1(\subset^{+}_{-}) \otimes Id_{Z_0(pt^+)}] \circ [\sigma(pt^-) \otimes \beta_{Z_0(pt^+),Z_1(pt^+)}] \circ [Z_0({}^{+}_{-}\supset) \otimes Id_{Z_1(pt^+)}]$$

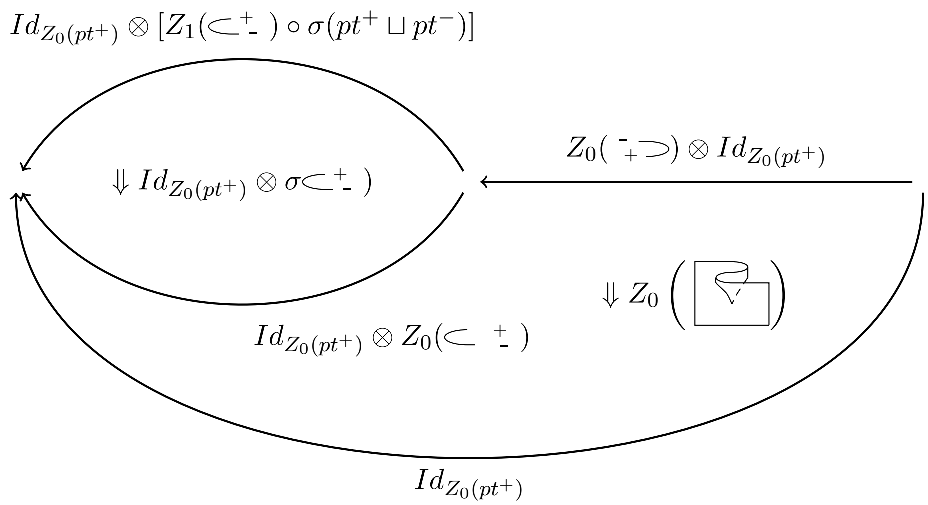
This composition can be visualized graphically.



Using the images of the folds and cusps under the TFTs Z_0 and Z_1, one can recover $\sigma(pt^-)$ from $\tilde{\sigma}(pt^+)$. Thus the first part of this data can be expressed as a pair of morphims,

$$\sigma : Z_0(pt^+) \rightleftarrows Z_1(pt^+) : \tilde{\sigma}$$

The rest of the data can, likewise, be expressed equivalently in terms of $\sigma(pt^+)$ and $\tilde{\sigma}$. For example, using the image under Z_0 of one of the cusps, we can form the following 2-morphism $\varepsilon = \tilde{\sigma}(\subset^{+}_{-})$.



It is perhaps more illuminating to view this morphism by representing it as a string diagram (see Appendix B.4), as in Figure 4.8. Using the cusps and relations from $\mathsf{Bord}_2^{\mathrm{or}}$, we can recover $\tilde{\sigma}(\subset^{+}_{-})$ from $\tilde{\sigma}(\subset^{+}_{-})$, and so it is equivalent data defining the transformation. Similarly, the 2-morphism $\sigma({}^{+}_{-}\supset)$ is equivalent data to a 2-morphism $\eta = \tilde{\sigma}({}^{+}_{-}\supset)$ which fills the pasting diagram of Figure 4.9.

We may now express the additional relations that σ must satisfy in terms of this new equivalent data. These relations express the naturality of the assignment $f \mapsto \sigma_f$, for 1-bordisms f. Thus for each 2-morphsims $\alpha : f \to g$, we get a relation which expresses the naturality with respect to α. It is enough to just check naturality for each of the generating 2-morphisms. In the oriented case there are roughly two groups of generating 2-morphisms. There are the invertible 2-morphisms, which are the cusps, and the non-invertible 2-morphisms, which are the four Morse generators.

Proposition 4.5.1. *Naturality with respect to the cusp 2-morphisms is equivalent to the following relations:*

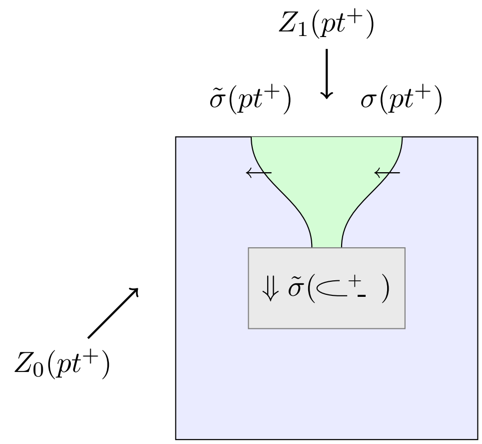
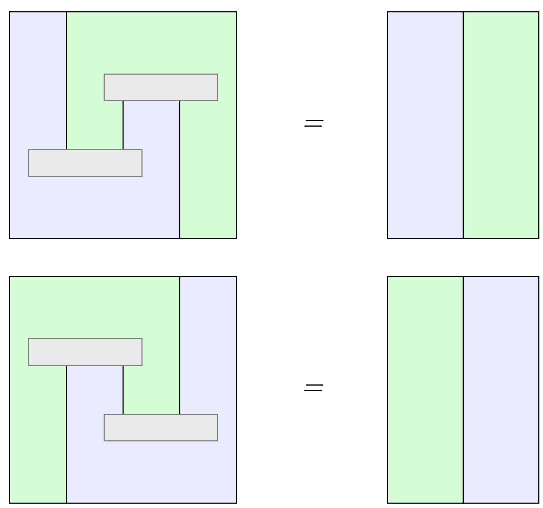


Figure 4.8: A Pasting Diagram For Transformations of TFTs



In particular, these form an adjunction between $Z_0(pt^+)$ and $Z_a(pt^+)$.

Proof. This follows directly by writing out the equations which express the naturality with respect to the cusp morphisms, one merely needs to identify both sides of the equation. □

Transformations, as defined in Definition B.1.6, have invertible 2-morphisms data. However, up to this point we have not used this fact in any way. It is interesting to understand what happens when we don't impose this invertibility constraint, i.e. when we consider lax and oplax transformations, see Remark B.1.7. As we will see shortly, any lax or oplax transformation of oriented (and hence unoriented) topological field theories

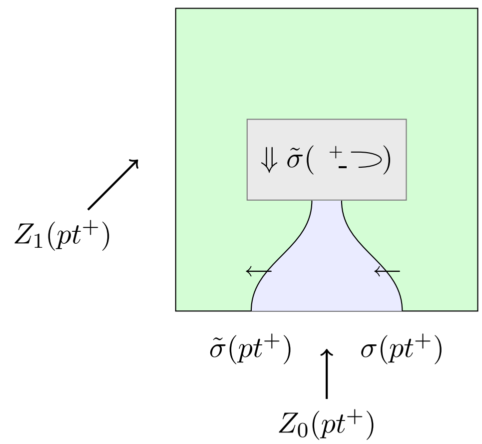
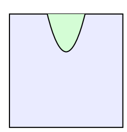
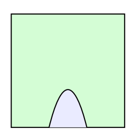


Figure 4.9: A second Pasting Diagram For Transformations of TFTs

automatically satisfies the required invertibility condition; any lax or oplax transformation is automatically a *strong* transformation.

Proposition 4.5.1 gives us justification for introducing a graphic notation which naturally expresses these relations. Compare with Appendix B.4.

$\varepsilon =$  $\eta =$ 

This graphical notation can also be extended to the original data describing the transformation σ. In this notation the naturality with respect to the cusp looks like the diagram in Figure 4.10.

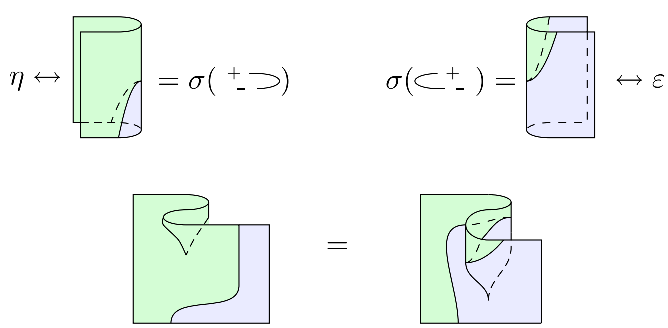


Figure 4.10: Naturality of Transformations with respect to a Cusp Generator

One of the most fascinating aspects of the bordism bicategory is how much duality is present. The generating 1-morphisms, the cusp 2-morphism, and their relations exhibit a duality between the positive and negative points. We have seen one aspect of this, namely that a lax transformation of field theories gives rise to a right-adjunction $\sigma : Z_0(pt^+) \to Z_1(pt^+)$. However there is much more structure. The 2D Morse generating 2-morphisms, together with the 2D Morse relations realize that fact that the two generating 1-morphisms are adjoint to one another. In fact they show that the generating 1-morphisms are *simultaneously* left-adjoint and right-adjoint. This simultaneous left/right adjunction is called an *ambidextrous adjunction*.

The significance of this for transformations of topological field theories is that such transformations are defined in terms of certain data, certain 1-morphism and 2-morphism. The 2-morphism have source and targets which are 1-morphisms which have adjoints. Consequently, as we have seen in Appendix B.4, because these generating 1-morphisms have left and right adjoints, we get a host of associated 2-morphisms: the mates. For example Figure 4.11 shows the mate of $\sigma(\subset^{+}_{-})$.

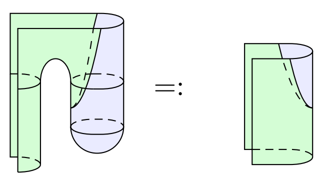
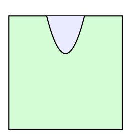
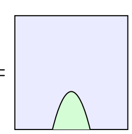


Figure 4.11: A Mate for a Transformation of Field Theories

Just as we turned the structure 2-morphisms for a transformation into 2-morphisms involving just the positive point, we can likewise turn the mates of these structure 2-morphisms into 2-morphisms involving just the positive point. These can also be given a graphical notation:

$\varepsilon^* =$  $\eta^* =$ 

It is an amusing exercise to show that this graphical notation is also justified, and that these mates give the data of a *left* adjunction $\sigma : Z_0(pt^+) \to Z_1(pt^+)$. Thus the data of any transformation of topological field theories automatically gives rise to an ambidextrous

adjunction, just from requiring naturality with respect to the cusp morphisms and by taking mates.

Finally we must require naturality with respect to the 2D Morse generators. Naturality with respect to these generators will force the mates to be the inverses of η and ε so that we actually have an adjoint equivalence between $Z_0(pt^+)$ and $Z_a(pt^+)$. In Figure 4.12 we show what naturality with respect to the cup generator means in terms of the graphical notation. We also show how this can equivalently be expressed as a relation among η, ε, η^*, and ε^*. Similarly we may translate the condition of being natural with respect each of the 2D Morse generators into relations among η, ε, η^*, and ε^*. These are summarized in Table 4.12. The two relations coming from the cusps are not independent. In fact they are equivalent, as the graphical calculation in Figure 4.13 shows.

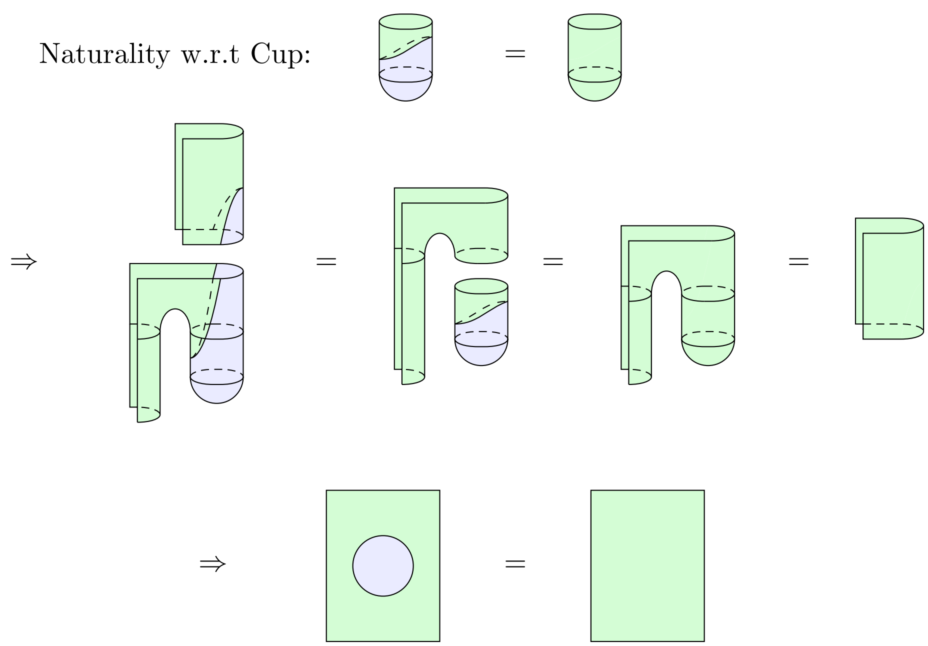
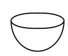
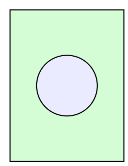
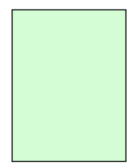
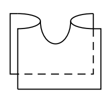
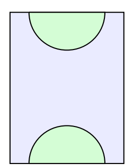
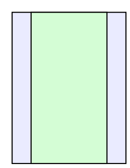
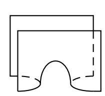
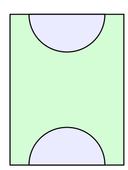
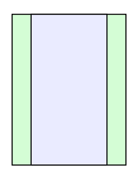
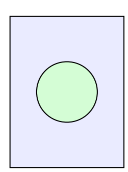
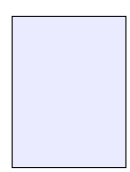


Figure 4.12: Naturality with respect to Cup Implies a Relation

Table 4.1: Relations Imposed on Transformations by Naturality with respect to Morse Generators

Generator	Naturality Relation		
		=	
		=	
		=	
		=	

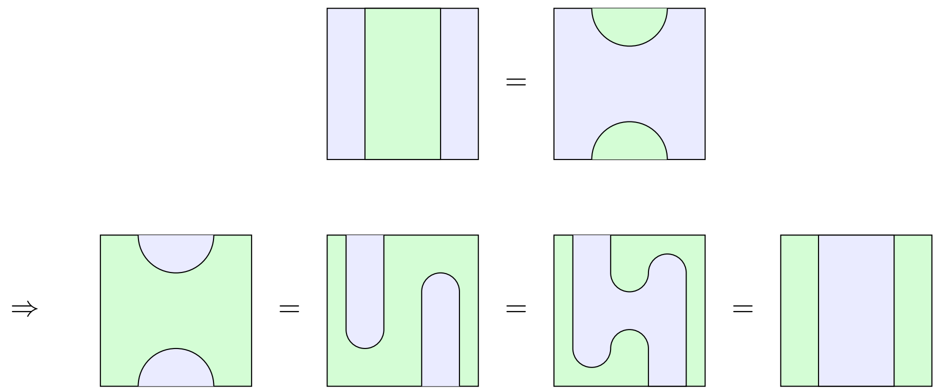


Figure 4.13: Relations from Cusps are not Independent

4.6 Examples: Extendeded TFTs = Separable Symmetric Frobenius Algebras

The classification results of Theorems 4.3.1 and 4.4.1 state that Bord_2 and $\mathsf{Bord}_2^{\mathrm{or}}$ are equivalent as a symmetric monoidal bicategories to particular symmetric monoidal bicategories, B and B^{or}, defined in terms of generators and relations. If C is an target symmetric monoidal bicategory, then the bicategory $\mathsf{SymBicat}(\mathsf{Bord}_2, \mathsf{C})$ is equivalent to the bicategory $\mathsf{SymBicat}(\mathsf{B}, \mathsf{C})$, and similarly in the oriented case. Using Propositions 3.5.7 and 3.6.1, we may characterize these bicategories completely. In this section we do this in the case where $\mathsf{C} = \mathsf{Alg}_k^2$, the bicategory of k-algebras, bimodules, and intertwiners.

We begin with some algebraic preliminaries. Consider Alg_k^2. The objects of this bicategory are k-algebras, the 1-morphisms are bimodules, and the 2-morphisms are bimodule maps (a.k.a. intertwiners). The composition of 1-morphisms is given by the tensor product of bimodules. There are two possible conventions, but for definiteness let us consider a B-A-bimodule, ${}_BM_A$, as a morphism from A to B. Equivalence in Alg_k^2 is called *Morita equivalence*, and an adjoint equivalence $({}_SM_R, {}_RN_S, \eta, \varepsilon)$ (see Definition B.4.2) is called a *Morita context*, see [Lam99].

Given an algebra homomorphism $h : A \to B$ we can turn it into a B-A-bimodule, ${}_BBh_A$. This bimodule is ${}_BB$ as a B-module. The right A-action comes from the right B-action and the homomorphism h. These bimodules are compatible with the composition

of homomorphisms,

$$ {}_C(Ch_2) \otimes_B (Bh_1)_A \cong {}_C(Ch_2h_1)_A. $$

In fact these isomorphisms can be chosen to be coherent, i.e. they give a homomorphism from the category of k-algebras and algebra homomorphisms (viewed as a bicategory with only identity 2-morphisms) to the bicategory Alg_k^2.

The tensor product over k gives Alg_k^2 the structure of a symmetric monoidal bicategory. In fact, it equips the category of algebras and homomorphisms with the structure of a symmetric monoidal category, and most of the structure of a symmetric monoidal bicategory can be transfered to Alg_k^2 via the above homomorphism. The remaining structure deals with the tensor product (over k) of bimodules and is easily supplied. We leave the details to the reader.

Recall from Definitions A.2.1 and A.2.2 the notion of a symmetric Frobenius algebra (A, e, λ). Here $e = \sum x_i \otimes y_i \in A \otimes_k A$ is a bicentral element and $\lambda : A \to k$ is a k-linear map, which satisfies the Frobenius normalization condition. A bicentral element $e \in A \otimes A$ is equivalent to a map of bimodules,

$$ {}_{A_1}A_{A_2} \otimes {}_{A_3}A_{A_4} \to {}_{A_1}A_{A_4} \otimes {}_{A_3}A_{A_2} $$

where we have labeled the algebras with numbers to keep track of the different actions. This correspondence arises because the source bimodule is cyclic as an $(A\otimes A)$-$(A\otimes A)$-bimodule with cyclic element $1 \otimes 1$. Hence determined by the image $e \in A \otimes A$ of $1 \otimes 1$, moreover it is easy to check that this element is bicentral. Expressed as a bimodule map, as above, it is clear that the bicentral element e can be transfered along Morita contexts. We will show that λ may also be transfered along Morita contexts.[1]

The linear map $\lambda : A \to k$ necessarily factors through the quotient,

$$ A \to A/[A, A] \to k, $$

where $[A, A]$ denotes the minimal k-submodule containing all elements of the form $a \cdot b - b \cdot a$. We have the following lemma:

Lemma 4.6.1. *Let A and B be k-algebras and $f = ({}_BM_A, {}_AN_B, \eta, \varepsilon)$ a Morita context. Then there is a canonical isomorphism of k-modules:*

$$ f_* : A/[A, A] \cong B/[B, B]. $$

[1] It is essential that we are considering *symmetric* Frobenius algebras. In general the property of being a Frobenius algerba is *not* Morita invariant, see [Lam99].

In particular the k-linear map λ may be transfered along Morita contexts.

Proof. We have the following sequence of canonical isomorphisms:

$$\begin{aligned}
A/[A,A] &\cong A \otimes_{A\otimes A^{\mathrm{op}}} A \\
&\cong (N \otimes_B M) \otimes_{A\otimes A^{\mathrm{op}}} (N \otimes_B M) \\
&\cong (M \otimes_A N) \otimes_{B\otimes B^{\mathrm{op}}} (M \otimes_A N) \\
&\cong B \otimes_{B\otimes B^{\mathrm{op}}} B \\
&\cong B/[B,B].
\end{aligned}$$

□

Definition 4.6.2. Let (A, e^A, λ^A) and (B, e^B, λ^B) be symmetric Frobenius algebras. Let $f = ({}_BM_A, {}_AN_B, \eta, \varepsilon)$ be a Morita context between A and B. Then we say that f is *compatible* with the symmetric Frobenius algebra structure if $f_*e^A = e^B$ and $f_*\lambda^A = \lambda^B$. ◇

Definition 4.6.3. Let Frob be the symmetric monoidal bicategory whose objects are separable symmetric Frobenius algebras, whose 1-morphisms are compatible Morita contexts, and whose 2-morphisms are isomorphisms of Morita contexts. ◇

Theorem 4.6.4. *The bicategory of 2-dimensional* oriented *extended topological field theories with values in* Alg^2 *is equivalent to the bicategory* Frob.

We will prove this theorem in the course of this section. Given a bimodule ${}_AM_B$, we can reinterpret the algebra actions and obtain a new bimodule ${}_{B^{\mathrm{op}}}\underline{M}_{A^{\mathrm{op}}}$. $\underline{M}$ has the same underlying k-module structure as M. If $f = ({}_BM_A, {}_AN_B, \eta, \varepsilon)$ is a Morita context, then $\underline{f} = ({}_{A^{\mathrm{op}}}\underline{M}_{B^{\mathrm{op}}}, {}_{A^{\mathrm{op}}}\underline{N}_{B^{\mathrm{op}}}, \eta, \varepsilon)$ is a Morita context between B^{op} and A^{op}.

Suppose now that $B = A^{\mathrm{op}}$. Then $\underline{M}$ and M are *both* A-A^{op}-bimodules. It then makes sense to consider bimodule maps from M to $\underline{M}$, and in particular isomorphisms. Similarly, if f is a Morita context from A to A^{op}, then so is $\underline{f}$.

Definition 4.6.5. A *stellar algebra* is an algebra A, equipped with a Morita context between A and A^{op}, $s = ({}_{A^{\mathrm{op}}}M_A, {}_AN_{A^{\mathrm{op}}}, \eta, \varepsilon)$, together with an isomorphism of Morita contexts, $\sigma : s \cong \underline{s}$. ◇

Example 4.6.6. An (algebraic) $*$-algebra gives rise to a stellar algebra. The $*$-structure on A is the same as an isomorphism of algebras, $* : A \to A^{\mathrm{op}}$. Then ${}_AA{*}_{A^{\mathrm{op}}}$ and ${}_{A^{\mathrm{op}}}A^{\mathrm{op}}{*}_A^{-1}$,

with the canonical unit and counit, form a Morita context s between A and A^{op}. The identity $*^2 = id$, then induces an isomorphism $s \cong \underline{s}$. ◇

Example 4.6.7. Not every stellar algebra arises from a $*$-structure. Let R be an algebra such that R is not isomorphic to $S = R^{\mathrm{op}}$, and such that R is also not isomorphic to $M_2(R)$.[2] Let $A = M_2(R) \oplus S$. We have $A^{\mathrm{op}} = M^2(S) \oplus R$, so that $A \ncong A^{\mathrm{op}}$, and hence A cannot support a $*$-structure. Nevertheless, A can be made into a stellar algebra in a straightforward manner. We leave the details as an exercise. ◇

Stellar structures can be transfered via Morita contexts, just as symmetric Frobenius algebra structures. Specifically, if $f = ({}_BM_A, {}_AN_B, \eta, \varepsilon)$ is a Morita context between A and B and $(A, s, \sigma : s \cong \underline{s})$ is a stellar structure on A, then $(B, f_*s, f_*\sigma)$ is a stellar structure on B, where $f_*s = \underline{f} \circ s \circ f$ and $f_*\sigma = \underline{f} \circ \sigma \circ f$.

Definition 4.6.8. Let $(A, s^A, \sigma^A : s^A \cong \underline{s}^A)$ and $(B, s^B, \sigma^B : s^B \cong \underline{s}^B)$ be two stellar algebras. A 1-morphism of stellar algebras (from A to B) consists of a Morita context f(between A and B) together with an isomorphism of Morita contexts:

$$\phi : f_*s^A \to s^B.$$

A 2-morphism between 1-morphisms (f, ϕ) and (f', ϕ') is an isomorphism of Morita contexts $\alpha : f \to f'$ such that the following triangle of isomorphism of Morita contexts commutes:

$$\begin{tikzcd}
f_*s^A \arrow[d, "\alpha"'] \arrow[dr, "\phi"] & \\
f'_*s^A \arrow[r, "\phi'"'] & s^B
\end{tikzcd}$$

◇

The stellar algebras, 1-morphism, and 2-morphisms form a symmetric monoidal bicategory in the obvious way.

Definition 4.6.9. Let Frob^* be the symmetric monoidal bicategory of separable stellar symmetric Frobenius algebras, that is the objects consist of quintuples $(A, s, \sigma, e, \lambda)$, where

[2]For example if k is a field whose Brauer group contains elements of order greater then two, then it will have algebras of this kind.

A is separable, (A, s, σ) is a stellar algebra, and (A, e, λ) is a symmetric Frobenius algebra. The 1-morphism of Frob^* consist of those 1-morphisms (f, ϕ) of stellar algebras for which f is compatible with the symmetric Frobenius algebra structure. The 2-morphism of Frob^* are the 2-morphisms of stellar algebras. ◇

Theorem 4.6.10. *The bicategory of 2-dimensional unoriented extended topological field theories with values in* Alg^2 *is equivalent to the bicategory* Frob^**.*

We will also prove this theorem in the course of this section. There is a symmetric monoidal homomorphism $\mathsf{Bord}_2^{\mathrm{or}} \to \mathsf{Bord}_2$, which forgets the orientation. Thus any unoriented topological field theory also gives rise to an oriented field theory. It make sense, then, to begin with the oriented classification and determine what additional structure is necessary to extend an oriented theory to an unoriented theory.

By 3.5.7 and 3.6.1 the bicategory $\mathsf{SymBicat}(\mathsf{B}^{\mathrm{or}}, \mathsf{Alg}^2)$ is equivalent to the bicategory of generating data in Alg^2 which satisfy the relations of Theorem 4.4.1. In particular there are two generating object corresponding to the two different oriented points. Their images in Alg^2 will correspond to two algebras $A = Z(pt^+)$ and $B = Z(pt^-)$.

Similarly there are two generating 1-morphisms which give rise to two bimodules

$$Z({}^{+}_{-}\!\supset) = {}_{A\otimes B}M$$
$$Z_1(\subset\!{}^{+}_{-}) = N_{A\otimes B}$$

As before these give rise to bimodules ${}_A\tilde{M}_{B^{\mathrm{op}}}$ and ${}_{B^{\mathrm{op}}}\tilde{N}_A$. The 2-morphism generators in the oriented case fall into two categories: the cusp generators and the 2D Morse generators.

The cusp generators are by far the easiest to recognize. While they naturally arise as morphisms involving M and N, we may equivalently express them in terms of $\tilde{M}$ and $\tilde{N}$. The four cusp generators then give rise to four bimodule maps,

$$f_1 : {}_A\tilde{M} \otimes_{B^{\mathrm{op}}} \tilde{N}_A \to {}_A A_A$$
$$f_2 : {}_A A_A \to {}_A\tilde{M} \otimes_{B^{\mathrm{op}}} \tilde{N}_A$$
$$f_3 : {}_{B^{\mathrm{op}}}\tilde{N} \otimes_A \tilde{M}_{B^{\mathrm{op}}} \to {}_{B^{\mathrm{op}}}B^{\mathrm{op}}{}_{B^{\mathrm{op}}}$$
$$f_4 : {}_{B^{\mathrm{op}}}B^{\mathrm{op}}{}_{B^{\mathrm{op}}} \to {}_{B^{\mathrm{op}}}\tilde{N} \otimes_A \tilde{M}_{B^{\mathrm{op}}}$$

The cusp inversion relations imply that f_1 and f_2 are inverses and f_3 and f_4 are inverses. The swallowtail relation implies that $(\tilde{M}, \tilde{N}, f_1, f_4)$ forms an adjoint equivalence, i.e. a

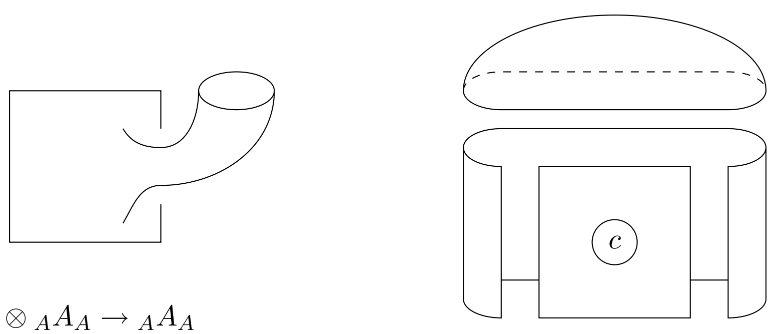
Morita context. Thus this portion of the topological field theory is equivalent to specifying a pair of algebras and a Morita context between them. We need to identify the remaining structures, which we do in a sequence of lemmas.

Lemma 4.6.11. *Let $A = Z(pt^+)$ be the vale of the positive point in an oriented topological field theory. There is a canonical isomorphism $A/[A, A] \cong z(A)$, where $z(A)$ is the center of the algebra A.*

Proof. Let $V = A/[A, A]$. Recall that the bimodule homomorphisms ${}_AA_A \to {}_AA_A$ can be canonically identified with the center $z(A)$. Gluing the basic generators as in Figure 4.14 and evaluating via the field theory provides maps

$$V \otimes {}_AA_A \to {}_AA_A$$
$$z(A) \to V$$

and the first of these is equivalent to a map $V \to Z(A)$. Here we have represented an element $c \in z(A)$ graphically as an endomorphism of ${}_AA_A$. The remaining figures in Figure 4.15 and Figure 4.16 show that these maps are in fact inverses. The first equality in Figure 4.16 follows by writing the surface in terms of the elementary bordisms and using standard manipulations available in any bicategory. □



$V \otimes {}_AA_A \to {}_AA_A$

$z(A) \to V$

Figure 4.14: Maps Between the Value of the Circle and the Center.

Remark 4.6.12. A similar calculation shows that the multiplication on V, induced from the "pants" bordism of Figure 1.1 agrees with the multiplication on $z(A)$. ◇

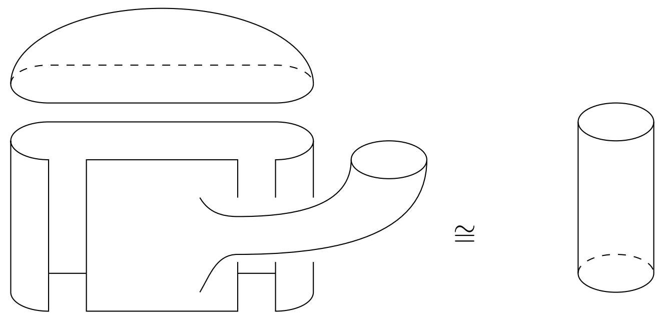


Figure 4.15: The Value of the Circle is the same as the Center, Part 1.

The cusp flip relations imply that the value of either of the two saddles determines the other. The value of the saddle under the topological field theory is equivalent to a bimodule map,

$$ {}_{A_1}A_{A_2} \otimes {}_{A_3}A_{A_4} \to {}_{A_1}A_{A_4} \otimes {}_{A_3}A_{A_2}. $$

Here we have labeled the various algebras with numbers to keep track of where each algebra acts. We have seen that since these bimodules are cyclic, such a map is determined by the image of a cyclic generator, for example $1 \otimes 1$. Thus this map is equivalent to a bicentral element $e = \sum a_i \otimes b_i \in A \otimes A$.

Lemma 4.6.13. *A is a separable algebra.*

Proof. As we have seen in Figures 4.14, 4.15 and 4.16, the saddle can be used to give a map $V \otimes A \to A$. In terms of the element e, this map is given by:

$$ [x] \otimes y \mapsto \sum_i y a_i x b_i. $$

As we have seen this map also induces and isomorphism between V the center $z(A)$, via $[x] \in V \leftrightarrow \sum a_i x b_i \in z(A)$. Since this map is an isomorphism, there exists an element $[z]$ such that $\sum a_i z b_i = 1 \in z(A)$. Thus $\sum a_i \otimes (z b_i)$ gives A the structure of a separable algebra. Letting z range over all possible representatives allows us obtain all the possible separability structures, and so we are left with the property that A is separable. $\square$

Lemma 4.6.14. *The element e given by the saddle, together with the cup morphism*

$$ \lambda : A \to A/[A, A] = V \to k $$

equip A with the structure of a symmetric Frobenius k-algebra.

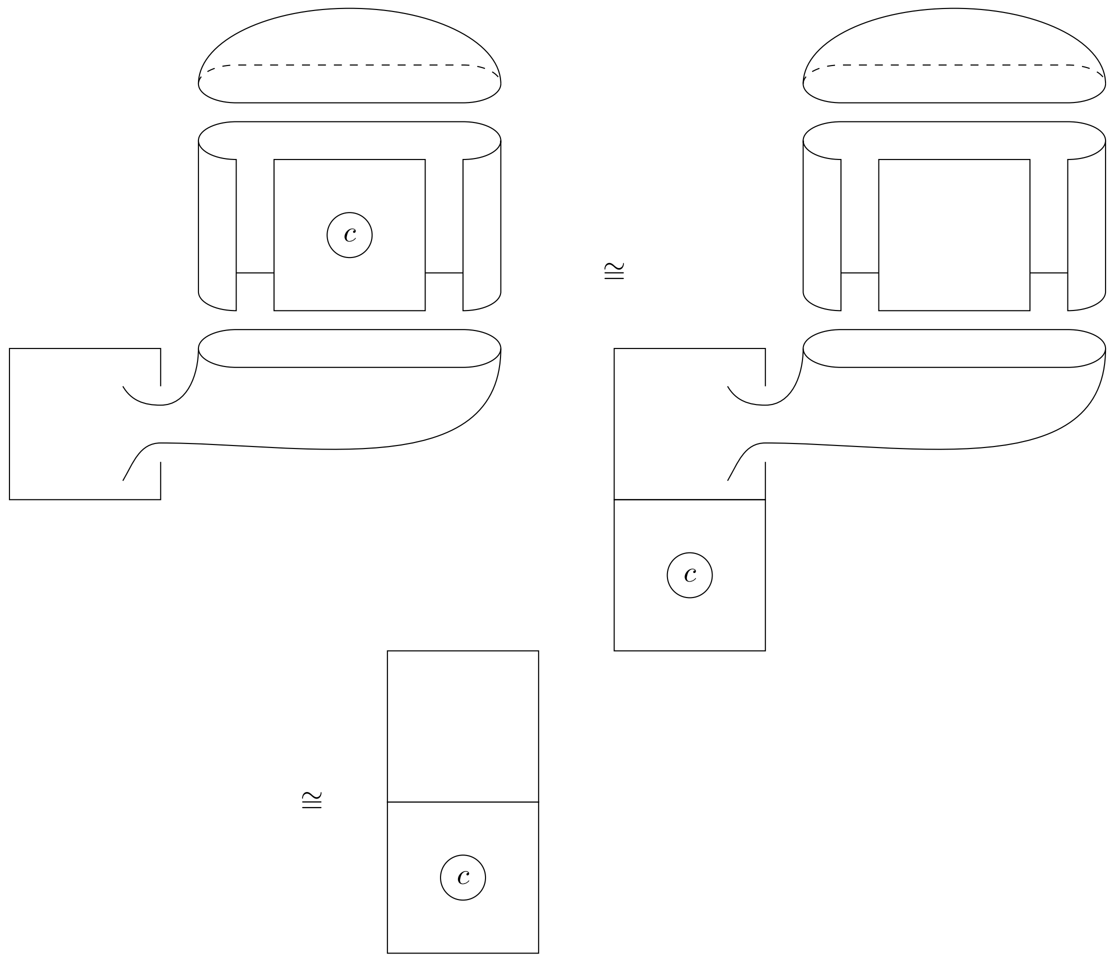


Figure 4.16: The Value of the Circle is the same as the Center, Part 2.

Proof. Since λ factors through $A/[A, A]$, if these structures make A into a Frobenius algebra, then A will be symmetric. We must show that the Frobenius equations hold. The Frobenius equations are precisely the 2D Morse relations in disguise. □

This completes most of the analysis in the oriented case. We will return to the oriented case momentarily, after making a brief digression on the unoriented case. In the unoriented case, there is no distinction between the positive and negative points. Thus the Morita context between A and B^{op} becomes a Morita context between A and A^{op}. The symmetry generators and relations, which are only relevant in the unoriented case, turn this Morita context into a stellar structure for A. This completes the proof of Theorem 4.6.10.

In the oriented case, the algebras A and B may be distinct. We have translated all the structure into structures on A. The structures on B are derived from these using the Morita context between A and B^{op}. However, the algebra B is actually irrelevant. The bicategory of separable symmetric Frobenius algebras A, together with an algebra B and a Morita equivalence $A \simeq B^{op}$ is equivalent to the bicategory which forgets the algebra B and the Morita equivalence. This is a straightforward exercise which we leave to the reader. This completes the proof of Theorem 4.6.4.

Corollary 4.6.15. *If $Z : \mathsf{Bord}_d \to \mathsf{Alg}^2$ is any bicategorical d-dimensional topological field theory and M is any $(d-2)$-manifold, then the algebra $Z(M)$ is separable. In particular over a field $Z(M)$ is semi-simple.*

Proof. There is a symmetric monoidal homorphism $(-) \times M : \mathsf{Bord}_2 \to \mathsf{Bord}_d$, which sends pt to M. Thus Z induces a 2-dimensional topological field theory $\hat{Z}$, in which $\hat{Z}(pt) = Z(M)$. $\square$

Appendix A

Algebras

A.1 Algebras and Duality

The following material is fairly standard are standard. Throughout, let k denote a fixed commutative ring. All algebras considered here will be algebras over k.

Definition A.1.1. An object, P, of an abelian category is *projective* if it satisfies the following universal lifting property: Given a surjection $p : A \twoheadrightarrow B$, and a map $f : P \to B$, there is at least one map $h : P \to A$ such that $f = p \circ h$.

$$\begin{tikzcd} & A \arrow[d, two heads, "p"] \\ P \arrow[ur, dashed, "\exists h"] \arrow[r, "f"'] & B \end{tikzcd}$$

In particular this defines projective modules in the categories Mod_R and ${}_R\mathsf{Mod}$, where R is a (possibly non-commutative) algebra. An R-module (left or right) is *finitely generated* if it admits a surjective map from a finite rank free module. ◇

Example A.1.2. Free modules are projective. ◇

Lemma A.1.3. *Let $M \in \mathsf{Mod}_R$ be an R-module. If the identity $M \to M$ factors as a map of R-modules,*

$$M \to (R)^{\times n} \to M$$

through a free module of rank $n < \infty$, then M is a finitely generated projective module.

Proof. M is finitely generated by definition. Let $A \twoheadrightarrow B$ be a surjection and let $f : M \to B$ be a map of R-modules. Since free modules are projective, we have a lift,

$$\begin{tikzcd} & & & A \arrow[d, two heads, "p"] \\ M \arrow[r, "g"] & (R)^n \arrow[urr, dashed, "h"] \arrow[r] & M \arrow[r, "f"'] & B \end{tikzcd}$$

The map $h \circ g : M \to A$ gives the necessary lift of f. ∎

The category ${}_R\mathsf{Mod}$ has a natural action by the symmetric monoidal category ${}_k\mathsf{Mod}$, given by the tensor product, $({}_RM, {}_kV) \mapsto M \otimes_k V$. Let R and S be algebras, and let M be an S-R-bimodule. Then $F = M \otimes_R (-) : {}_R\mathsf{Mod} \to {}_S\mathsf{Mod}$ defines a functor which is linear with respect to the ${}_k\mathsf{Mod}$-action in the sense that there is a natural isomorphism of functors $F(N \otimes_k V) \cong F(N) \otimes_k V$ for all $V \in {}_k\mathsf{Mod}$. These natural isomorphisms may be chosen to be coherent.

Lemma A.1.4. *Let R and S be two algebras and let $F : {}_R\mathsf{Mod} \to {}_S\mathsf{Mod}$ be a functor which*

1. *Is ${}_k\mathsf{Mod}$-linear in the above sense, and*

2. *Preserves coequalizers.*

Then $F \cong M \otimes_R (-)$ for some S-R-bimodule M.

Proof. Let $N \in {}_R\mathsf{Mod}$ be a left R-module. We have the following coequalizer sequence in ${}_R\mathsf{Mod}$:

$${}_RR \otimes_k R \otimes_k N \rightrightarrows {}_R \otimes_k N \to {}_RN,$$

where the two left-most maps are given by the action of R on itself and the action of R on N, respectively. Since F preserves coequalizers, we have the following is a coequalizer sequence in ${}_S\mathsf{Mod}$:

$$F(R \otimes_k R \otimes_k N) \rightrightarrows F(R \otimes_k N) \to F(N).$$

Since F is ${}_k\mathsf{Mod}$-linear, this is the same as the coequalizer,

$${}_SF(R) \otimes_k R \otimes_k N \rightrightarrows {}_SF(R) \otimes_k N \to {}_SF(N).$$

Thus ${}_SF(N) \cong {}_SF(R) \otimes_R N$ as S-modules, and so $F \cong M \otimes_R (-)$ for the S-R-bimodule $M = F(R)$. ∎

Similar results hold for right modules, as well. For our purposes, the most important application of this is when F is a left-adjoint, ${}_k\mathsf{Mod}$-linear functor. In this case the above theorem implies that F comes from a bimodule.

Lemma A.1.5. *Let M be an S-R-bimodule. If ${}_SM \otimes_R (-)$ is a right-adjoint, then M_R is a finitely generated projective R-module. If $(-) \otimes_S M_R$ is a right-adjoint, then ${}_SM$ is a finitely generated projective S-module.*

Proof. We prove this for left-modules. The case of right-modules is similar. Let $F = {}_SM \otimes_R (-)$. If F is a right-adjoint, then there exists a functor G, together with natural transformations $\eta : id \to GF$ and $\varepsilon : FG \to id$, realizing the adjunction. Equivalently, we have a natural isomorphism,

$$\mathrm{Hom}_S(G(A), B) \cong \mathrm{Hom}_R(A, F(B))$$

for all $A \in {}_R\mathsf{Mod}$ and $B \in {}_S\mathsf{Mod}$.

F is ${}_k\mathsf{Mod}$-linear since it arises from a bimodule. In fact G is also ${}_k\mathsf{Mod}$-linear, which can be seen as follows. For any $V \in {}_k\mathsf{Mod}$, $A \in {}_R\mathsf{Mod}$, and $B \in {}_S\mathsf{Mod}$, we have,

$$\begin{aligned}
\mathrm{Hom}_S(G(A \otimes_k V), B) &\cong \mathrm{Hom}_R(A \otimes_k V, F(B)) \\
&\cong \mathrm{Hom}_k(V, \mathrm{Hom}_R(A, F(B))) \\
&\cong \mathrm{Hom}_k(V, \mathrm{Hom}_S(G(A), B)) \\
&\cong \mathrm{Hom}_S(G(A) \otimes_k V, B).
\end{aligned}$$

Since this is true for all B, by the Yoneda lemma $G(A \otimes_k V) \cong G(A) \otimes_k V$, so that G is ${}_k\mathsf{Mod}$-linear. The previous lemma now implies that $G = {}_RN \otimes_S (-)$ for some R-S-bimodule, N.

The counit and unit of the adjunction satisfy the equations,

$$\begin{aligned}
id_F &= \varepsilon F \circ F\eta, \\
id_G &= G\varepsilon \circ \eta G.
\end{aligned}$$

Writing these out in terms of the bimodules ${}_SM_R$ and ${}_RN_S$ we see that ε and η become bimodule maps,

$$\begin{aligned}
\varepsilon &: {}_RN \otimes_S M_R \to {}_RR_R, \\
\eta &: {}_SS_S \to {}_SM \otimes_R N_S.
\end{aligned}$$

Since S is cyclic as an $S \otimes S^{\mathrm{op}}$-module, η is determined by the single element $\eta(1) = \sum_{i \in I} x_i \otimes_R y_i \in M \otimes_R N$. As an aside, $\eta(1)$ is *central* in the sense that for all $s \in S$, we have $\sum (s x_i) \otimes_R y_i = \sum x_i \otimes_R (y_i s)$. The adjunction equation may now be written as,

$$\sum_i \varepsilon(a \otimes_S x_i) y_i = a,$$
$$\sum_i x_i \varepsilon(y_i \otimes_S b) = b,$$

for all $a \in N$ and $b \in M$.

The identity map on M may now be factored through a finite rank free R-module, as follows. Define maps:

$$M_R \to (R_R)^I \qquad\qquad (R_R)^I \to M_R$$
$$m \mapsto (\varepsilon(y_i, m))_{i \in I} \qquad\qquad (r_i)_{i \in I} \mapsto \sum_i x_i r_i$$

The adjunction equations imply that the composition is the identity on M_R. Since I is finite, Lemma A.1.3 ensures that M_R is a finitely generated projective R-module. $\square$

In the above we call the bimodule ${}_R N_S$ the *left dual* of the bimodule ${}_S M_R$. Notice that the analogous argument (using the remaining adjunction equation) shows that N is a finitely generated projective S-module.

A.2 Symmetric Frobenius Algebras

Definition A.2.1. A *Frobenius algebra* is triple (A, λ, e) where A is a k-algebra, $\lambda : A \to k$ is a k-linear functional, $e = \sum_i x_i \otimes y_i \in A \otimes_k A$ is A-*central*, i.e. for all $w \in A$, $\sum_i (w \cdot x_i) \otimes y_i = \sum_i x_i \otimes (y_i \cdot w)$, and such that the following Frobenius normalization condition is satisfied,

$$\sum_i \lambda(x_i) y_i = \sum_i x_i \lambda(y_i) = 1_A.$$

$\diamond$

Definition A.2.2. A Frobenius algebra (A, λ, e) is *symmetric* if λ is trace-like, i.e. $\lambda(xy) = \lambda(yx)$ for all $x, y \in A$. $\diamond$

Definition A.2.3. Let V be a k-module. A bilinear form, $b : V \otimes V \to k$ is *non-degenerate* if the following conditions are satisfied:

1. $b(x,y) = 0$ for all $x \in V$ implies $y = 0$,
2. $b(x,y) = 0$ for all $y \in V$ implies $x = 0$.

The k-*dual* of A is defined to be $\hat{A} := \mathrm{Hom}_k(A,k)$. It is naturally an A-A-bimodule with actions,

$$a \cdot f \cdot b \mapsto (x \mapsto f(bxa)).$$

$\diamond$

There are several equivalent formulations of the notion of a Frobenius algebra.

Proposition A.2.4. *Given a fixed k-algebra A, the following structure/property combinations are equivalent:*

1. *(λ, e) such that (A, λ, e) forms a Frobenius algebra.*
2. *(b, e) where $e \in A \otimes A$ is A-central, $b : A \otimes A \to k$ is a non-degenerate k-bilinear form such that $b(xy,z) = b(x,yz)$ and such that (A, λ, e) is a Frobenius algebra, where $\lambda(x) = b(x, 1_A)$.*
3. *(b, e), where $b : A \otimes A \to k$ is a non-degenerate bilinear form satisfying $b(xy,z) = b(x,yz)$ and $e : k \to A \otimes A$ is an A-central element, such that the* snake relations *are satisfied:*
$$(id_A \otimes b) \circ (e \otimes id_A) = (b \otimes id_A) \circ (Id_A \otimes e) = id_A.$$
4. *$b : A \otimes A \to k$, a non-degenerate bilinear form satisfying $b(xy,z) = b(x,yz)$ together with the property that A is a finitely generated projective k-module.*
5. *$\lambda : A \to k$ such that* $\ker \lambda$ *contains no nontrivial left ideals, together with the property that A is a finitely generated projective k-module. Left ideals may be replaced by right ideals to yield an equivalent characterization.*
6. *$b : A \otimes A \to k$, a non-degenerate bilinear form satisfying $b(xy,z) = b(x,yz)$ and such that b induces an isomorphism of right A-modules $A_A \cong \hat{A}_A$, together with the property that A is a finitely generated projective k-module. Again, left-modules may be used instead. (A k-module A which is isomorphic to $\hat{A}$ is called* reflexive*).*
7. *An isomorphism of right A-modules $A_A \cong \hat{A}_A$, (or an isomorphism of left A-modules ${}_AA \cong {}_A\hat{A}$), together with the property that A is a finitely generated projective k-module*

8. (Δ, λ) *where* $\Delta : A \to A \otimes_k A$ *is a comultiplication with counit* $\lambda : A \to k$ *such that the* Frobenius equations *are satsified*

$$\Delta \circ m = (m \otimes id_A) \circ (id_A \otimes \Delta) = (id_A \otimes m) \circ (\Delta \otimes id_A)$$

where $m : A \otimes_k A \to A$ *is the multiplication map.*

9. (Δ, λ) *where* $\Delta : A \to A \otimes_k A$ *is a comultiplication with counit* $\lambda : A \to k$ *such that* $\Delta : {}_AA_A \to {}_AA \otimes A_A$ *is a bimodule map.*

Proof. The equivalences (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) are fairly standard and are left as an exercise for the reader. It is equally clear that (8) $\Leftrightarrow$ (9). (2) $\Leftrightarrow$ (3), once one realizes that the snake equation and the Frobenius equation are equivalent in this context. By construction (2) $\Rightarrow$ (1) and (1) $\Rightarrow$ (2) as follows: letting $b = \lambda \circ m$ yields a bilinear pairing compatible with the multiplication in the necessary way. We need only show that b is non-degenerate. Suppose that there exists an element $z \in A$ such that $0 = b(w, z) = \lambda(w \cdot z)$ for all $w \in A$. Then,

$$z = z \cdot 1_A = z \cdot \sum_i x_i \lambda(y_i) = \sum_i (z \cdot x_i)\lambda(y_i) = \sum_i x_i \lambda(y_i \cdot z) = 0$$

Thus b is non-degerate on the right. A similar calculation shows b is non-degenerate on the left. (9) $\Rightarrow$ (1) by taking $e = \Delta(1_A)$. The fact that Δ is a bimodule map implies that e is A-central and the compatibility of comultiplication and counit yield the Frobenius equations. Finally, (2) $\Rightarrow$ (9) by taking $\Delta(z) := \sum_i x_i \otimes (y_i \cdot z)$. This is a bimodule map since e is A-central, and is a comultiplication for which λ is the counit by virtue of the Frobenius equations.

Thus it remains to prove that the (4) to (7) group is equivalent to the (1), (2), (3), (8), (9) group. (3) implies that A is a finitely generated projective module and hence (3) $\Rightarrow$ (4). The reverse implication follows by choosing a basis $a_i \in A$, i.e. the images of the standard basis via some surjective map $(k)^n \to A$. Such a basis exists since A is finitely generated. Moreover, since A is finitely generated and projective, and more specifically b induces an isomorphism $A \cong \hat{A}$, we may define elements $\hat{a}_i \in A$, by the following equation,

$$b(\hat{a}_i, a_j) = \delta_{ij}.$$

Thus we have the following identities, $\sum a_i b(\hat{a}_i, x) = x = \sum b(x, a_j)\hat{a}_j$. Define the element $e = a_i \otimes \hat{a}_i \in A \otimes A$. By construction, e satisfies the Frobenius normalization condition. It is straightforward to check that e is A-central. $\square$

The conditions in (6) and (7) that A be a finitely generated k-module are necessary as the following example shows. This example was pointed out to me by D. Eisenbud.

Example A.2.5. Let L be a field and consider $k = L[x, y, z]$. Let $f : k^3 \to k$ be defined by the matrix (x, y, z). There is a short exact sequence,

$$0 \to K \to k^3 \xrightarrow{f} k$$

Now k has projective dimension 3, so K cannot be a projective module. This is actually part of the larger Koszul complex,

$$0 \to k \xrightarrow{g} k^3 \to k^3 \xrightarrow{f} k$$

Which allows us split off the following short exact sequence,

$$0 \to k \xrightarrow{g} k^3 \to K \to 0$$

We apply $Hom_k(-, k)$ to get,

$$0 \to Hom_k(K, k) \to Hom_k(k^3, k) \to Hom_k(k, k)$$

Which becomes

$$0 \to Hom_k(K, k) \to k^3 \xrightarrow{g^*} k$$

Here g^* is the matrix $(x, -y, z)$, so this sequence is actually exact on the right as well. We find that $Hom_k(K, k) \cong K$. So K is a reflexive k-module, which is finitely generated, but is not projective. $\diamond$

Proposition A.2.6. *If (A, λ, e) a Frobenius algebra, then the property of being symmetric is equivalent to the following characterizations.*

1. *λ is trace-like.*

2. *The bilinear form b is symmetric.*

3. *The element e is A-bicentral, i.e. for all $w, z \in A$ we have,*

$$\sum_i (w \cdot x_i \cdot z) \otimes y_i = \sum_i x_i \otimes (z \cdot y_i \cdot w)$$

4. *The isomorphism $\hat{A} \cong A$ induced by b is an isomorphism ${}_A A_A \cong {}_A \hat{A}_A$ of bimodules.*

Proof. (1) $\Leftrightarrow$ (2) is clear, from the proof of the previous lemma. (2) $\Rightarrow$ (4), as follows. The image in $\hat{A}$ of the element a is the assignment $x \mapsto b(a, x)$ and so we must show that $b(waz, x) = b(a, zxw)$ for all $z, w, x, a \in A$. A direct calculation, assuming b is symmetric, shows:

$$\begin{aligned} b(waz, x) &= b(wa, zx) \\ &= b(zx, wa) \\ &= b(zxw, a) \\ &= b(a, zxw). \end{aligned}$$

Similarly (3) $\Rightarrow$ (1), as follows. If $e = \sum x_i \otimes y_i$ is bicentral, then, by the snake relations we have,

$$\begin{aligned} \lambda(ab) &= \lambda(\sum x_i \lambda(y_i a) b) \\ &= \sum \lambda(y_i a)\lambda(x_i b) \\ &= \sum \lambda(b y_i)\lambda(a x_i) \\ &= \lambda(\sum b \lambda(a x_i) y_i) \\ &= \lambda(ba). \end{aligned}$$

Finally, (4) $\Rightarrow$ (3) as follows. By assumption we have $b(waz, x) = b(a, zxw)$ for all $z, w, x, a \in A$. Thus,

$$\begin{aligned} \sum_i (z x_i w) \otimes y_i &= \sum_{i,j} x_j b(y_j, z x_i w) \otimes y_i \\ &= \sum_{i,j} x_j \otimes b(w y_j z, x_i) y_i \\ &= \sum_j x_j \otimes (w y_j z), \end{aligned}$$

so that e is, indeed, bicentral. □

A.3 Fully-Dualizable Algebras and Separable Algebras

Definition A.3.1. A k-algebra A is *separable* algebra if there exists an A-central element $\tilde{e} = \sum_i \tilde{x}_i \otimes \tilde{y}_i \in A \otimes A$ such that the separability normalization condition is satisfied,

$$\sum_i x_i y_i = 1_A.$$

◇

Remark A.3.2. In this formulation, a Frobenius algebra is an algebra equipped with additional structure, whereas a separable algebra is an algebra satisfying a property. A separable symmetric Frobenius algebra is a Frobenius algebra which happens to be separable. ◇

The following results can be found in [DI71]. Let $A^e = A \otimes_k A^{op}$ be the *enveloping algebra* of A. As A is an A-A-bimodule, it may equivalently be considered a (left) A^e-module. The multiplication map $\mu : A^e \to A$ is an A^e-module map with kernel J, the left ideal of A^e generated by elements of the form $a \otimes 1 - 1 \otimes a$. Note an element $e \in A \otimes A$ is A-central precisely when, viewed as an element in A^e, is we have $Je = 0$.

Proposition A.3.3. *For a k-algebra A the following properties are equivalent,*

1. *A is separable, that is there exists $\tilde{e} \in A^e$ such that $J\tilde{e} = 0$ and $\mu(\tilde{e}) = 1$.*

2. *A is projective as a left A^e-module under the μ-structure.*

3. *The exact sequence of left A^e-modules,*

$$0 \to J \to A^e \xrightarrow{\mu} A \to 0$$

   splits.

If in addition, k is a field then we may add,

4. *A is finite dimensional and* classically separable*, that is for every field extension K of k we have that $A \otimes_k K$ is semi-simple.*

In particular A is semi-simple in this case. If k is a perfect field (for example a characteristic zero field or a finite field), then any finite dimensional semi-simple algebra is separable.

Definition A.3.4 (Lurie)**.** A k-algbera A is *fully-dualizable* if

1. It is separable,

2. It is finitely generated and projective as a k-module.

◇

The above definition can be extracted from J. Lurie's recent expository paper [Lur09]. This second condition is part of being a Frobenius algebra, and thus we see that a separable symmetric Frobenius algebra is the same as a fully-dualizable algebra with a non-degenerate symmetric trace, λ.

Appendix B

Bicategories

B.1 Bicategories, Homomorphisms, Transformations and Modifications

Definition B.1.1. A *bicategory* B consists of the following data,

1. A collection ob$\mathsf{B}$ whose elements are called *objects*. (We denote objects $a, b, c \in \mathsf{B}$. Sometimes the collection ob$\mathsf{B}$ is written B_0).

2. Categories $\mathsf{B}(a, b)$ for each pair of objects $a, b \in \mathsf{B}$. The objects of $\mathsf{B}(a, b)$ are referred to as *1-morphisms* from a to b (which we denote collectively as B_1, with elements $f, g, h, \dots$). The morphisms of $\mathsf{B}(a, b)$ are referred to as *2-morphisms* (which we denote collectively by B_2, with elements $\alpha, \beta, \gamma, \dots$).

3. Functors:

$$\begin{aligned} c_{abc} : \mathsf{B}(b, c) \times \mathsf{B}(a, b) &\to \mathsf{B}(a, c) \\ (g, f) &\mapsto g \circ f \\ (\beta, \alpha) &\mapsto \beta * \alpha \end{aligned}$$

   and $I_a : \mathbf{1} \to \mathsf{B}(a, a)$, for all objects $a, b, c \in \mathsf{B}$, where $\mathbf{1}$ denotes the singleton category (thus the functor I_a is equivalent to specifying an *identity element* I_a). The functors c are called the *horizontal compositions*.

4. Natural isomorphisms:

$$a : c_{abd} \circ (c_{bcd} \times id) \to c_{acd} \circ (id \times c_{abc})$$
$$\ell : c_{abb} \circ (I_b \times id) \to id$$
$$r : c_{aab} \circ (id \times I_a) \to id$$

known, respectively, as the *associator* and left and right *unitors*.

(thus invertible 2-morphisms $a_{h,g,f} : (h \circ g) \circ f \to h \circ (g \circ f)$, $\ell_f : I_b \circ f \to f$ and $r_f : f \circ I_a \to f$).

These are required to satisfy the pentagon and triangle identities:

The Pentagon Identity

$$\begin{tikzcd}
&& (kh)(gf) \arrow[drr, "a"] && \\
((kh)g)f \arrow[urr, "a"] \arrow[d, "a*1"'] &&&& k(h(gf)) \\
(k(hg))f \arrow[rrrr, bend right, "a"'] &&&& k((hg)f) \arrow[u, "1*a"']
\end{tikzcd}$$

The Triangle Identity

$$\begin{tikzcd}
(gI)f \arrow[rr, "a"] \arrow[dr, "r*1"'] && g(If) \arrow[dl, "1*\ell"] \\
& gf &
\end{tikzcd}$$

A bicategory in which the associators and left and right unitors are identities, so that composition is strictly associative, is called a *2-category*. A bicategory A is *pointed* if it is equipped with a distinguished object $p \in \mathsf{A}$. $\diamond$

Warning B.1.2. The horizontal composition of 1-morphism is denoted $f \circ g$, which is contradictory to the notation for 2-morphisms. The vertical composition of 2-morpisms is denoted $\alpha \circ \beta$, while the horizontal composition is denoted $\alpha * \beta$. Regrettably this is standard notation, and so we reluctantly comply. ⌟

Let A be a bicategory and let $\sigma : a \to b$ be a 1-morphism in A. Pre- and post-composition define functors,

$$\sigma^* : \mathsf{A}(b,c) \to \mathsf{A}(a,c)$$
$$\sigma_* : \mathsf{A}(d,a) \to \mathsf{A}(d,b).$$

The associators define canonical natural isomorphisms for each 1-morphism $\theta \in \mathsf{A}_1$.

$$\sigma_* \circ \theta_* \cong (\sigma \circ \theta)_*$$
$$\sigma^* \circ \theta^* \cong (\theta \circ \sigma)^*$$
$$\sigma^* \circ \theta_* \cong \theta_* \circ \sigma^*$$

provided the corresponding compositions are allowed.

Definition B.1.3 (Internal Equivalence)**.** Let B be a bicategory. An *equivalence* in B is a 1-morphism $f : a \to b$ in $\mathsf{B}(a,b)$ such that there exists a 1-morphism $g : b \to a$ in $\mathsf{B}(b,a)$ with isomorphisms $\eta : I_a \to g \circ f$ in $\mathsf{B}(a,a)$ and $\varepsilon : f \circ g \to I_b$ in $\mathsf{B}(b,b)$. In this case we say that a is *equivalent* to b. ◇

Remark B.1.4. It is clear that the relation of equivalence is an equivalence relation on the objects of B. The set of equivalence classes is denoted $\pi_0\mathsf{B}$. ◇

Definition B.1.5. Let A and B be bicateogries. A *homomorphism* $F : \mathsf{A} \to \mathsf{B}$ consists of the data:

1. A function $F : \mathrm{ob}\ \mathsf{A} \to \mathrm{ob}\ \mathsf{B}$,
2. Functors $F_{ab} : \mathsf{A}(a,b) \to \mathsf{B}(F(a), F(b))$,
3. Natural isomorphisms

$$\phi_{abc} : c^{\mathsf{B}}_{F(a)F(b)F(c)} \circ (F_{bc} \times F_{ab}) \to F_{ac} \circ c^{\mathsf{A}}_{abc}$$
$$\phi_a : I^{\mathsf{B}}_{F(a)} \to F_{aa} \circ I^{\mathsf{A}}_a$$

(thus invertible 2-morphisms $\phi_{gf} : Fg \circ Ff \to F(g \circ f)$ and $\phi_a : I^{\mathsf{B}}_{Fa} \to F(I^{\mathsf{A}}_a)$).

such that the following diagrams commute:

$$\begin{tikzcd}
 & (Fh \circ Fg) \circ Ff \arrow[dl, "a^{\mathsf{B}}"'] \arrow[dr, "\phi * 1_{Ff}"] & \\
Fh \circ (Fg \circ Fh) \arrow[d, "1_{Fh} * \phi"'] & & F(h \circ g) \circ Ff \arrow[d, "\phi"] \\
Fh \circ F(g \circ f) \arrow[dr, "\phi"'] & & F((h \circ g) \circ f) \arrow[dl, "Fa^{\mathsf{A}}"] \\
 & F(h \circ (g \circ f)) &
\end{tikzcd}$$

$$\begin{tikzcd}
 & (Ff) \circ (I^{\mathsf{B}}_{Fb}) \arrow[dl, "1_{Ff} * \phi_b"'] \arrow[dr, "r^{\mathsf{B}}"] & & (I^{\mathsf{B}}_{Fa}) \circ (Ff) \arrow[dl, "\ell^{\mathsf{B}}"'] \arrow[dr, "\phi_a * 1_{Ff}"] & \\
(Ff) \circ (FI^{\mathsf{A}}_b) \arrow[dr, "\phi"'] & & Ff & & (FI^{\mathsf{A}}_a) \circ (Ff) \arrow[dl, "\phi"] \\
 & F(f \circ I\mathsf{A}_b) \arrow[ur, "Fr^{\mathsf{A}}"'] & & F(I^{\mathsf{A}}_a \circ f) \arrow[ul, "F\ell^{\mathsf{A}}"] &
\end{tikzcd}$$

If the natural isomorphisms ϕ_{abc} and ϕ_a are identities, then the homomorphism F is called a *strict homomorphism*. A homomorphism F between pointed bicategories $(\mathsf{A}, p_{\mathsf{A}})$ and $(\mathsf{B}, p_{\mathsf{B}})$ is a *pointed homomorphism* if $F(p_{\mathsf{A}}) = p_{\mathsf{B}}$. $\diamond$

Definition B.1.6. Let $(F, \phi), (G, \psi) : \mathsf{A} \to \mathsf{B}$ be two homomorphisms between bicategories. A *transformation* $\sigma : F \to G$ is given by the data:

1. 1-morphisms $\sigma_a : Fa \to Ga$ for each object $a \in \mathsf{A}$,
2. Natural Isomorphisms, $\sigma_{ab} : (\sigma_a)^* \circ G_{ab} \to (\sigma_b)_* \circ F_{ab}$

   (thus invertible 2-morphisms $\sigma_f : Gf \circ \sigma_a \to \sigma_b \circ Ff$ for every $f \in \mathsf{A}_1$).

such that the diagrams in Figure B.1 commute for all 1-morphisms in A, $f : a \to b$ and $g : b \to c$. If A, B, (F, ϕ) and (G, ψ) are pointed, then the transformation σ is a *pointed transformation* if $\sigma_{p_{\mathsf{A}}} = I_{p_{\mathsf{B}}}$. $\diamond$

$$
\begin{tikzcd}
& G(g \circ f) \circ \sigma_a \arrow[dr, "\sigma_{gf}"] & \\
(Gg \circ Gf) \circ \sigma_a \arrow[ur, "\psi_{g,f} * id_{\sigma_a}"] \arrow[d, "a^{\mathsf{B}}"'] & & \sigma_c \circ F(g \circ f) \\
Gg \circ (Gf \circ \sigma_a) \arrow[d, "id_{Gg} * \sigma_f"'] & & \sigma_c \circ (Fg \circ Ff) \arrow[u, "id_{\sigma_c} * \phi_{g,f}"'] \\
Gg \circ (\sigma_b \circ Ff) \arrow[dr, "(a^{\mathsf{B}})^{-1}"'] & & (\sigma_c \circ Fg) \circ Ff \arrow[u, "a^{\mathsf{B}}"'] \\
& (Gg \circ \sigma_b) \circ Ff \arrow[ur, "\sigma_g * id_{Ff}"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& \sigma_a \arrow[dr, "(r^{\mathsf{B}})^{-1}"] & \\
I^{\mathsf{B}}_{Ga} \circ \sigma_a \arrow[ur, "\ell^{\mathsf{B}}"] \arrow[d, "\psi_a * id_{\sigma_a}"'] & & \sigma_a \circ I^{\mathsf{B}}_{Fa} \\
(GI^{\mathsf{A}}_a) \circ \sigma_a \arrow[rr, "\sigma_{I^{\mathsf{A}}_a}"'] & & \sigma_a \circ (FI^{\mathsf{A}}_a) \arrow[u, "id_{\sigma_a} * \phi_a^{-1}"']
\end{tikzcd}
$$

Figure B.1: Transformation Axioms

Remark B.1.7. There is a weaker notion of *lax transformation* in which the natural transformations σ_f are not assumed to be isomorphisms. This imposes an obvious directional bias. An *oplax transformation* is defined similarly, but with the direction of σ_f reversed. When comparing these notions of transformation, the transformations, as defined above with invertible 2-morphism data, are called *strong* transformations. $\diamond$

Definition B.1.8. Let $(F, \phi), (G, \psi) : \mathsf{A} \to \mathsf{B}$ be two homomorphisms between bicategories and let $\sigma, \theta : F \to G$ be two transformations between homomorphisms. A *modification* $\Gamma : \sigma \to \theta$ consists of 2-morphisms $\Gamma_a : \sigma_a \to \theta_a$ for every object $a \in \mathsf{A}$, such that the following square commutes:

$$\begin{tikzcd}
Gf \circ \sigma_a \arrow[r, "id * \Gamma_a"] \arrow[d, "\sigma_f"'] & Gf \circ \theta_a \arrow[d, "\theta_f"] \\
\sigma_b \circ Ff \arrow[r, "\Gamma_b * id"'] & \theta_b \circ Ff
\end{tikzcd}$$

for every 1-morphism $f : a \to b$ in A. If A, B, (F, ϕ), (G, ψ), σ and θ are pointed, then we say Γ is a *pointed modification* if $\Gamma_{p_{\mathsf{A}}}$ is the identity of $I_{p_{\mathsf{B}}}$. $\diamond$

Definition B.1.9. Let (F, ϕ), (G, ψ) and (H, κ) be homomorphisms from the bicategory A to the bicategory B. Let $\sigma : F \to G$ and $\theta : G \to H$ be two transformations. We define the composition $\theta \circ \sigma$ to be the transformation $F \to H$ given by the following data:

1. The 1-morphism $\theta_a \circ \sigma_a$ for each object $a \in \mathsf{A}$,

2. The natural isomorphism $(\theta_a \circ \sigma_a)^* \circ H_{ab} \to (\theta_a \circ \sigma_a)_* \circ F_{ab}$ defined by the following sequence of natural isomorphisms:

$$\begin{tikzcd}
(\theta_a \circ \sigma_a)^* \circ H_{ab} \arrow[dr] & & \sigma_a^* \circ (\theta_b)_* \circ G_{ab} \arrow[dr] & & (\theta_b)_* \circ (\sigma_a)_* \circ F_{ab} \arrow[dr] & \\
& \sigma_a^* \circ \theta_a^* \circ H_{ab} \arrow[ur, "\theta"] & & (\theta_b)_* \circ \sigma_a^* \circ G_{ab} \arrow[ur, "\sigma"'] & & (\theta_a \circ \sigma_a)_* \circ F_{ab}
\end{tikzcd}$$

where the unlabeled arrows are the canonically defined natural transformations induced by the associators of B.

Given transformations $\sigma, \sigma', \sigma'' : F \to G$ and modifications $\Gamma : \sigma \to \sigma'$ and $\Sigma : \sigma' \to \sigma''$, the *vertical composition* of Γ and Σ is the modification $\Sigma \circ \Gamma : \sigma \to \sigma''$ given by the 2-morphisms $\Sigma_a \circ \Gamma_a$ for each object $a \in \mathsf{A}$.

Similarly, if (F, ϕ), (G, ψ) and (H, κ) are homomorphisms from the bicategory A to the bicategory B, $\sigma, \sigma' : F \to G$ and $\theta, \theta' : G \to H$ are transformations and $\Gamma : \sigma \to \sigma'$ and $\Sigma : \theta \to \theta'$ are modifications, then the *horizontal composition* of Γ and Σ is the modification $\Sigma * \Gamma : \theta \circ \sigma \to \theta' \circ \sigma'$ given by the 2-morphisms $\Sigma_a * \Gamma_a$ for each object $a \in \mathsf{A}$. $\diamond$

With these compositions we get a bicategory $\mathsf{Bicat}(\mathsf{A}, \mathsf{B})$ for each pair of bicategories A, B. The objects are the homomorphisms from A to B, the 1-morphisms are the transformations, and the 2-morphisms are the modifications. Compositions are as above and the associators and unitors are the obvious ones coming from B.

Remark B.1.10. Since the associators and unitors come from those in B, $\mathsf{Bicat}(\mathsf{A},\mathsf{B})$ is a 2-category whenever B is a 2-category. $\diamond$

If A and B are pointed bicategories, we can also consider the pointed homomorphisms, transformations and modifications. The vertical composition of two pointed modifications is again a pointed modification, but the horizontal composition of two pointed transformations or modifications fails to be pointed. However we can fix this. The problem can be seen at the level of transformations. If σ and θ are two compatible pointed transformations, as above, then the transformation $\theta \circ \sigma$ satisfies:

$$(\theta \circ \sigma)_{p_\mathsf{A}} = I_{p_\mathsf{B}} \circ I_{p_\mathsf{B}}$$

This is not equal to I_{p_B} and hence this is not a pointed transformation. However there is a canonical isomorphism:

$$I_{p_\mathsf{B}} \circ I_{p_\mathsf{B}} \to I_{p_\mathsf{B}}$$

given by the right or left unitor (they yield the same isomorphism). Using this isomorphism we can replace the transformation $(\theta \circ \sigma)$ with a pointed transformation $(\theta \circ \sigma)'$ as follows:

$$(\theta \circ \sigma)'_a = \begin{cases} (\theta \circ \sigma)_a & \text{if } a \neq p_\mathsf{A} \\ I_{p_\mathsf{B}} & \text{if } a = p_\mathsf{A} \end{cases}$$

$$(\theta \circ \sigma)'_f = \begin{cases} (\theta \circ \sigma)_f & \text{if } f : a \to b \text{ and } a, b \neq p_\mathsf{A} \\ (\theta \circ \sigma)_f \circ (\text{can.}) & \text{if } f : a \to b,\ a = p_\mathsf{A},\ \text{and } b \neq p_\mathsf{A} \\ (\text{can.}) \circ (\theta \circ \sigma)_f & \text{if } f : a \to b,\ b = p_\mathsf{A},\ \text{and } a \neq p_\mathsf{A} \\ (\text{can.}) \circ (\theta \circ \sigma)_f \circ (\text{can.}) & \text{if } f : a \to b,\ a = b = p_\mathsf{A} \end{cases}$$

where "(can.)" denotes the above canonical 2-morphisms. The reader can check that this is in fact a transformation. This is defines the pointed composition of pointed transformations. A similar construction yields the pointed horizontal composition of pointed modifications. Again we obtain a bicategory $\mathsf{Bicat}_*(\mathsf{A},\mathsf{B})$ of pointed homomorphism, pointed transformations, and pointed modifications. The canonical constant functor sending every object of A to p_B gives $\mathsf{Bicat}_*(\mathsf{A},\mathsf{B})$ the structure of a pointed bicategory.

These bicategories share an additional structure. There are composition homomorphisms,

$$\mathsf{Bicat}(\mathsf{B},\mathsf{C}) \times \mathsf{Bicat}(\mathsf{A},\mathsf{B}) \to \mathsf{Bicat}(\mathsf{A},\mathsf{C})$$

which have their own additional coherence structures. This can be summarized by saying there is a tricategory whose objects are bicategories and whose hom-bicategories are the bicategories $\mathsf{Bicat}(\mathsf{A}, \mathsf{B})$. We will not need the full details of these coherence structures, but the interested reader should consult [GPS95]. Actually there is some ambiguity in defining what the new composition of transformations and modifications should be. One way to address this is to introduce the operation of *whiskering*, an approach which we sketch here. This yields two different but canonical ways to compose transformations (and modifications), and hence results in two different tricategories. However, these two tricategories are equivalent as tricategories, again see [GPS95] for full details.

Definition B.1.11. Let $(F, \phi) : \mathsf{A} \to \mathsf{B}$ and $(G, \psi) : \mathsf{B} \to \mathsf{C}$ be two homomorphism between bicategories. We define their composition to be the homomorphism $G \circ F : \mathsf{A} \to \mathsf{C}$ given by the following data:

1. On objects we have the map $G \circ F : \mathrm{ob}\ \mathsf{A} \to \mathrm{ob}\ \mathsf{B}$,
2. Functors $G_{Fa,Fb} \circ F_{ab} : \mathsf{A}(a, b) \to \mathsf{A}(GF(a), GF(b))$,
3. Natural transformations which are the composite,

$$I_{GF(a)} \overset{\psi}{\to} G(I_{Fa}) \overset{G(\phi)}{\to} GF(I_a),$$

and

$$\begin{array}{c} c^{\mathsf{C}}_{GF(a)GF(b)GF(c)} \circ (G_{FbFc} \times G_{FaFb}) \circ (F_{bc} \times F_{ab}) \\ \big\downarrow \psi_{FaFbFc} \circ (F_{bc} \times F_{ab}) \\ G_{GF(a)GF(c)} \circ c^{\mathsf{B}}_{FaFbFc} \circ (F_{bc} \times F_{ab}) \\ \big\downarrow G_{GF(a)GF(c)} \circ \phi_{abc} \\ G_{FaFc} \circ F_{ac} \circ c^{\mathsf{A}}_{abc} \end{array}$$

which in components is given by:

$$GF(g) \circ GF(f) \overset{\psi}{\to} G(F(g) \circ F(f)) \overset{G\phi}{\to} GF(g \circ f).$$

$\diamond$

Remark B.1.12. The composition of pointed homomorphisms is again pointed. Notice also that the above composition of homomorphisms is strictly associative and hence gives rise to the ordinary category bicat whose objects are bicategories and whose morphisms are homomorphisms, see [Bén67]. ◇

Definition B.1.13 (Whiskering)**.** Let $\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}$ be bicategories and $(F, \phi^F) : \mathsf{A} \to \mathsf{B}$, $(G, \phi^G) : \mathsf{B} \to \mathsf{C}$, $(\bar{G}, \phi^{\bar{G}}) : \mathsf{B} \to \mathsf{C}$, and $(H, \phi^H) : \mathsf{C} \to \mathsf{D}$ be homomorphisms, and let $(\sigma_x, \sigma_f) : G \to \bar{G}$ be a transformation. We define the *pre-whiskering* of σ with F to be the transformation:

$$\sigma F = (\sigma_{F(a)}, \sigma_{F(f)}).$$

The *post-whiskering* of σ with H is similarly defined to be the transformation:

$$H\sigma = (H(\sigma_a), H(\sigma_f)).$$

Let $\theta = (\theta_x, \theta_f) : G \to \overline{G}$ be another transformation and let $\Gamma : \sigma \to \theta$ be a modification. The the *pre-whiskering* of Γ with F is the modification $\Gamma F : \sigma F \to \theta F$ given by components $\Gamma_{F(a)}$. The *post-whiskering* of Γ with H is the modification $H\Gamma : H\sigma \to H\theta$ given by components $H(\Gamma_x)$. ◇

Whiskering is a strictly associative operation. Suppose that $G, \bar{G} : \mathsf{A} \to \mathsf{B}$ and $H, \bar{H} : \mathsf{B} \to \mathsf{C}$ are homomorphisms of bicategories and that $\sigma : G \to \bar{G}$ and $\beta : H \to \bar{H}$ are transformations. There are two natural choices for defining the composition of σ and β as transformations between $H \circ G \to \bar{H} \circ \bar{G}$. First we can whisker σ by $\overline{H}$ and then β by G, these new transformations can then be composed in $\mathsf{Bicat}(\mathsf{A}, \mathsf{C})$. Alternatively, we can whisker σ by H and β by $\overline{G}$. These transformations can also be composed in $\mathsf{Bicat}(\mathsf{A}, \mathsf{C})$, see Figure B.2. A similar discussion applies to modifications. Generally these operations don't agree, but they yield equivalent tricategories, see [GPS95].

Definition B.1.14 (External Equivalence)**.** A homomorphism $F : \mathsf{A} \to \mathsf{B}$ is an *equivalence of bicategories* if there exists a homomorphism $G : \mathsf{B} \to \mathsf{A}$ and an equivalence $id_{\mathsf{B}} \simeq F \circ G$ in the bicategory $\mathsf{Bicat}(\mathsf{B}, \mathsf{B})$ and an equivalence $id_{\mathsf{A}} \simeq G \circ F$ in the bicategory $\mathsf{Bicat}(\mathsf{A}, \mathsf{A})$. ◇

Theorem B.1.15 (Whitehead's theorem for bicategories)**.** *A homomorphism $F : \mathsf{A} \to \mathsf{B}$ is an equivalence of bicategories if and only if*

1. *F induces an isomorphism $\pi_0\mathsf{A} \cong \pi_0\mathsf{B}$. (Essentially surjective on objects).*

$$
\begin{tikzcd} {} \arrow[r, "G"] & {} \arrow[r, bend left, "H"] \arrow[r, bend right, "\overline{H}"'] \arrow[r, phantom, "\Downarrow\beta"] & {} \\ {} \arrow[r, bend left, "G"] \arrow[r, bend right, "\overline{G}"'] \arrow[r, phantom, "\Downarrow\sigma"] & {} \arrow[r, "\overline{H}"'] & {} \end{tikzcd}
\qquad \text{or} \qquad
\begin{tikzcd} {} \arrow[r, bend left, "G"] \arrow[r, bend right, "\overline{G}"'] \arrow[r, phantom, "\Downarrow\sigma"] & {} \arrow[r, "H"] & {} \\ {} \arrow[r, "\overline{G}"'] & {} \arrow[r, bend left, "H"] \arrow[r, bend right, "\overline{H}"'] \arrow[r, phantom, "\Downarrow\beta"] & {} \end{tikzcd}
$$

Figure B.2: Two Compositions in the Tricategory Bicat.

2. *$F_{ab} : \mathsf{A}(a, b) \to \mathsf{B}(Fa, Fb)$ is essentially surjective for all $a, b \in \mathsf{A}$. (Essentially full on 1-morphism).*

3. *$F_{ab} : \mathsf{A}(a, b) \to \mathsf{B}(Fa, Fb)$ is fully-faithful for all $a, b \in \mathsf{A}$. (Fully-faithful on 2-morphisms).*

The proof of this theorem is a routine but tedious application of the axiom of choice. We prove a similar theorem in Section 3.4 and the proof of the above theorem can be extracted from there.

B.2 Coherence and Strictification

The following discussion is based on the exposition in [Mac71]. A *bracketing* consists of a parenthesized word in "$-$" (dashes) and "1" (ones). The *length* of a bracketing is the number of dashes. More precisely, we make the following recursive definition: Both (1) and $(-)$ are bracketings (of length zero and one, respectively). In u and v are bracketings, then $(u)(v)$ is a bracketing of length $\text{length}(u) + \text{length}(v)$. The examples below have lengths five and four respectively.

$$((--)1)(((--)1)(1-)) \qquad \text{and} \qquad ((((11)1)-)((--)1))-$$

Let A be a bicategory. Given a bracketing b of length n, and a word w of composable 1-morphism from A of length n, then composition in A gives a canonical 1-morphism in A, which we denote $b(w)$. Define the following elementary moves on bracketings:

1. (Deleting Identities) $1b \Leftrightarrow b$ and $b1 \Leftrightarrow b$

2. (Rebracketing) $(bb')b'' \Leftrightarrow b(b'b'')$

where b, b', b'' represent arbitrary bracketings. An elementary move has an apparent source and target which are bracketings. A *path* also has source and target bracketings and is defined recursively as follows:

1. Elementary moves are paths,
2. If p and p' are paths such that the source of p is the target of p', then $p \circ p'$ is a path with source the source of p' and target the target of p.
3. if p and p' are paths, then $(p)(p')$ is a path with source $(\text{source}(p))(\text{source}(p'))$ and target $(\text{target}(p))(\text{target}(p'))$.

Given two bracketings b and b' of length n and a word w of composable 1-morphism from A, also of length n, we have the two 1-morphisms $b(w)$ and $b'(w)$ of A. Given a path s starting at b and ending at b', we get a canonical 2-morphism of A, $s(w) : b(w) \to b(w')$, given by replacing the elementary moves by the appropriate unitors and associators from A.

Theorem B.2.1 (MacLane's Coherence Theorem [Mac71])**.** *If b and b' are two bracketings of length n and w is a word of composable 1-morphism from the bicategory A, also of length n, then there exists a path $s : b \to b'$ starting at b and ending at b'. Moreover, given any two such paths $s, s' : b \to b'$, the resulting 2-morphisms $s(w) : b(w) \to b(w')$ and $s'(w) : b(w) \to b(w')$ are identical.*

MacLane's coherence theorem allows us to form the following construction. Define the *standard bracketing* b^n_{std} of length n to be the bracketing:

$$((\cdots(((--)-)-)\cdots)-).$$

Given a bicategory A we define the following associated 2-category, $Q(A)$:

- The objects of $Q(\mathsf{A})$ are the same as those of A.
- The 1-morphisms of $Q(\mathsf{A})$ are the words of composable 1-morphisms in A.
- Given two such words w, w', the 2-morphisms are the 2-morphisms between $b^{|w|}_{\text{std}}(w)$ and $b^{|w'|}_{\text{std}}(w')$.

Vertical composition coincides with that in A. The horizontal composition of 1-morphisms is given by concatenating words, and hence is strictly associative. The horizontal composition of 2-morphisms is given by using the canonical rebracketing 2-morphisms supplied by MacLane's coherence theorem. Given two 2-morphisms $\alpha : b_{\mathrm{std}}(w) \to b'_{\mathrm{std}}(w')$ and $\beta : b^{|\tilde{w}|}_{\mathrm{std}}(\tilde{w}) \to b^{|\tilde{w}'|}_{\mathrm{std}}(\tilde{w}')$, their horiszontal composition is defined to be the composite:

$$b^{|w\tilde{w}|}_{\mathrm{std}}(w\tilde{w}) \to b^{|w|}_{\mathrm{std}}(w) \circ b^{|\tilde{w}|}_{\mathrm{std}}(\tilde{w}) \stackrel{\alpha*\beta}{\to} b^{|w'|}_{\mathrm{std}}(w') \circ b^{|\tilde{w}'|}_{\mathrm{std}}(\tilde{w}') \to b^{|w'\tilde{w}'|}_{\mathrm{std}}(w'\tilde{w}')$$

where the unlabeled arrows are the canonical 2-morphisms from MacLane's theorem.

This composition is automatically associative and hence defines a 2-category $Q(\mathsf{A})$, called the *strictification* of A. In fact Q defines a functor,

$$Q : \mathsf{bicat} \to 2\,\mathsf{cat}_s$$

where $2\,\mathsf{cat}_s$ is the category of 2-categories and strict homomorphisms. This is left-adjoint to the inclusion functor $i : 2\,\mathsf{cat}_s \to \mathsf{bicat}$ into bicategories and (weak) homomorphisms.

In particular the unit and co-unit of the adjunction give us for each 2-category B a strict homomorphism $Q(\mathsf{B}) \to \mathsf{B}$ (which we can take to be the homomorphism which sends a composable word of 1-morphisms in B to its composition) and for each bicategory A a weak homomorphism $\mathsf{A} \to Q(\mathsf{A})$ (which we can take to be the homomorphism which sends a 1-morphism in A to the singleton word consisting of exactly that 1-morphism). Both of these homomorphisms are equivalences of bicategories. Note, however, that for a 2-category B, the inverse equivalence to the canonical strict homomorphism $Q(\mathsf{B}) \to \mathsf{B}$ is nearly always a *weak* homomorphism.

Corollary B.2.2. *Every bicategory is equivalent to a 2-category.*

B.3 Symmetric, Braided and Monoidal Categories

Symmetric monoidal categories are pervasive in mathematics and historically played a vital role in the development of the theory of bicategories. It is no surprise that many theorems about monoidal categories have analogs for bicategories as well. The presentation given here is designed to emphasize this connection.

Definition B.3.1. A *monoidal category* is a bicategory with one object. $\diamond$

Thus a monoidal category consists of a category M (the category of 1-morphisms, and 2-morphisms) with a specfied 1-morphism $1 \in M$ and a horizontal composition functor,

$$c = \otimes : M \times M \to M,$$

together with associator and unitor natural isomorphism satisfying the pentagon and triangle identities. We will typically use the notation $(M, \otimes)$ for monoidal categories. The reader is invited to verify that the above definitions of monoidal category, monoidal functor, and monoidal natural transformation coincide with the standard definitions, as presented in [Mac71], for example.

One is now tempted to define morphisms and higher morphisms for monoidal categories by simply using those which already exist for bicategories, however, this does not recover the usual notion of monoidal functor nor monoidal natural transformation. Indeed, there would also be an additional categorical layer coming from the modifications. The solution to the problem is to realize that, as a bicategory, a monoidal category is canonically pointed, and thus it is natural to define morphisms of monidal categories to be *pointed* morphisms of bicategories.

Definition B.3.2. A *monoidal functor* from the monoidal category $(M, \otimes)$ to $(M', \otimes')$ is a pointed homomorphism between the corresponding bicategories. A *monoidal natural transformations* is a pointed transformation. ◇

All pointed modifications between monoidal categories are automatically trivial, so that the bicategory $\mathsf{Bicat}_*((M, \otimes), (M', \otimes'))$ is in fact just an ordinary category. This agrees with the usual notion of monoidal functor between monoidal categories and monoidal natural transformation between these.

There is an obvious notational danger with defining monoidal categories to be bicategories with a single object. Traditionally a monoidal category is thought of as a category with extra structure, and hence has objects and morphisms. However viewed as a bicategory, the objects become 1-morphisms, and the morphisms become 2-morphisms. Unfortunately this point of confusion is inevitable. We will try to stick with the terminology from the bicategorical perspective as much as possible.

Recall the following well known construction: Algebras form the objects of a bicategory, Alg^2, whose 1-morphisms are bimodules and whose 2-morphisms are bimodule maps. The horizontal composition is given by the tensor product of bimodules. Given a algebra

A, we can recover the center of A as the algebra of bimodule endomorphism of the identity 1-morphism ${}_A A_A$, i.e. $Z(A) = {}_A \operatorname{Hom}_A(A, A)$. Similarly in an arbitrary bicategory, B, one can define the center $Z(X)$ of an object X to be the commutative monoid $\mathsf{B}(I_X, I_X)$. The following is a higher categorical analog of this construction.

Definition B.3.3. Let $(M, \otimes)$ be a monoidal category. Let id_M be the identity homomorphism from M to M. Define the *center* $Z(M)$ of M to be the full sub-bicategory of $\mathsf{Bicat}(M, M)$ whose only object is id_M. That is $Z(M)$ is the bicategory whose only object is id_M, whose morphisms are the (non-pointed) transformations $id_M \to id_M$, and whose 2-morphisms are the modifications between these. The 1-morphisms of $Z(M)$ are called *half-braidings.* $\diamond$

Half-braidings were introduced by Izumi in the context of subfactors [Izu00]. By construction $Z(M)$ is a monoidal category. Notice that we did not use *pointed* transformations and modifications. Unpacking this definition, we see that a half-braiding consists of:

1. a 1-morphism $X \in M$,
2. a family of invertible 2-morphisms $\gamma_{Y,X} : Y \otimes X \to X \otimes Y$ where $Y \in M$ runs over all 1-morphisms.

These data are required to satisfy:

(HBR0) (Naturality) The $\gamma_{-,X}$ form a natural isomorphism of functors:

$$\gamma_{-,X} : (-) \otimes X \to X \otimes (-).$$

(HBR1) For all Y, Z, the following diagram commutes:

$$\begin{tikzcd}
 & (Z \otimes Y) \otimes X \arrow[r, "\gamma_{Z \otimes Y, X}"] \arrow[dl, "a"'] & X \otimes (Z \otimes Y) & \\
Z \otimes (Y \otimes X) \arrow[dr, "1 \otimes \gamma_{Y,X}"'] & & & (X \otimes Z) \otimes Y \arrow[ul, "a"'] \\
 & Z \otimes (X \otimes Y) \arrow[r, "a^{-1}"'] & (Z \otimes X) \otimes Y \arrow[ur, "\gamma_{Z,X} \otimes 1"'] &
\end{tikzcd}$$

(HBR2) The following diagram commutes:

$$\begin{tikzcd} 1 \otimes X \arrow[r, "\ell"] \arrow[rr, bend right, "\gamma_{1,X}"'] & X \arrow[r, "r^{-1}"] & X \otimes 1 \end{tikzcd}$$

A 2-morphism between half-braidings (say from $(X, \{\gamma_{-,X}\})$ to $(X', \{\gamma_{-,X'}\})$) consists of a 2-morphism $\Gamma : X \to X'$ in M such that for all $Y \in M$ the following diagram commutes:

$$\begin{tikzcd} Y \otimes X \arrow[r, "1 * \Gamma"] \arrow[d, "\gamma_{Y,X}"'] & Y \otimes X' \arrow[d, "\gamma_{Y,X'}"] \\ X \otimes Y \arrow[r, "\Gamma * 1"'] & X' \otimes Y \end{tikzcd}$$

There is a strict forgetful homomorphism $U : Z(M) \to M$ which sends a half-braiding, $(X, \{\gamma_{-,X}\})$, to its underlying 1-morphism, X.

Definition B.3.4. Let M be a monoidal category. A *braiding* for M is a strict section of the homomorphism $U : Z(M) \to M$. That is, a strict homomorphism $s : M \to Z(M)$ such that $U \circ s = 1_M$ (strict equality). $\diamond$

Thus a braiding for a monoidal category M consists of an assignment: for each X a family of 2-morphisims $\gamma_{Y,X}$ which are natural in the Ys, which satisfy several conditions. Among these conditions is the requirement that this assignment must be a homomorphism, and so must also be natural in the X variable. More precisely, we have the following characterization:

Proposition B.3.5. *Let M be a monoidal category. A braiding for M consists of 2-morphisms $\gamma_{X,Y} : X \otimes Y \to Y \otimes X$ for each pair of 1-morphisms X, Y, such that:*

(BR0) The γ form a natural transformation,

$$\gamma : \otimes \to \otimes \circ \tau$$

where $\tau : M \times M \to M \times M$ denotes the flip functor,

(BR1) For all X, Y, Z, the following diagrams commute:

$$\begin{tikzcd}
& (Z \otimes Y) \otimes X \arrow[r, "\gamma_{Z\otimes Y,X}"] \arrow[dl, "a"'] & X \otimes (Z \otimes Y) & \\
Z \otimes (Y \otimes X) \arrow[dr, "1\otimes\gamma_{Y,X}"'] & & & (X \otimes Z) \otimes Y \arrow[ul, "a"'] \\
& Z \otimes (X \otimes Y) \arrow[r, "a^{-1}"'] & (Z \otimes X) \otimes Y \arrow[ur, "\gamma_{Z,X}\otimes 1"'] &
\end{tikzcd}$$

(BR2) For all $X, Y, Z \in M$ the following diagram commutes:

$$\begin{tikzcd}
& Z \otimes (Y \otimes X) \arrow[r, "\gamma_{Z,Y\otimes X}"] \arrow[dl, "a^{-1}"'] & (Y \otimes X) \otimes Z & \\
(Z \otimes Y) \otimes X \arrow[dr, "\gamma_{Z,Y}\otimes 1"'] & & & Y \otimes (X \otimes Z) \arrow[ul, "a^{-1}"'] \\
& (Y \otimes Z) \otimes X \arrow[r, "a"'] & Y \otimes (Z \otimes X) \arrow[ur, "1\otimes\gamma_{Z,X}"'] &
\end{tikzcd}$$

Proof. The necessity of (BR0) has already been established. (BR1) is just (HBR1) from the definition of a half-braiding. (BR2) is precisely the expression that the section $s : M \to Z(M)$ is a strict homomorphism so that $s(Y \otimes X) = s(Y) \otimes s(X)$. Compare with Definition B.1.9. Given (BR0-BR2), then (HBR2) is automatically satisfied as is the condition that $s(1) = 1$, see [JS86]. □

Definition B.3.6. A *braided monoidal category* is a monoidal category equipped with a braiding. A monoidal functor (F, ϕ) between braided monoidal categories is a *braided monoidal functor* if it satisfies

$$\begin{tikzcd}
F(X) \otimes F(Y) \arrow[r, "\gamma_{FX,FY}"] \arrow[d, "\phi_{A,B}"'] & F(Y) \otimes F(X) \arrow[d, "\phi_{B,A}"] \\
F(X \otimes Y) \arrow[r, "F(\gamma_{X,Y})"'] & F(Y \otimes X)
\end{tikzcd}.$$

A braided monoidal category is *symmetric* if the braiding satisfies $\gamma_{X,Y} \circ \gamma_{Y,X} = 1_{Y \otimes X}$. $\diamond$

B.4 Pastings, Strings, Adjoints, and Mates

Bicategories have two kinds of compositions, horizontal and a vertical, and this makes them inherently two-dimensional entities. For this reason many properties, structures, and equations are best expressed in a corresponding 2-dimensional formalism. In ordinary category theory one might specify certain structures on a category (for example, in 1-dimensional terms, the functor $\otimes$ and natural transformations a, ℓ, r of a monoidal category). These structures can then be combined in a one-dimensional, linear fashion. Often one imposes conditions on such a structure such as requiring that two or more different linear compositions must give the same result. Typical examples are the pentagon and triangle identities for monoidal categories.

For bicategories, due to their two-dimensional nature, structures and properties are best expressed in two-dimensional terms. Structures for bicategories, such as those that would give us a symmetric monoidal bicategory (the subject of Chapter 3), often require that two or more different two-dimensional compositions of morphisms must be the same. There are two common equivalent notations for describing these two-dimensional compositions: pasting diagrams and string diagrams. We will describe both here.

Pasting diagrams are a natural extension of the sort of diagrams commonly drawn in ordinary category theory. Typical diagrams in ordinary category theory consist of some vertices (labeled by the objects of the category being considered) together with directed edges (which are labeled by morphisms of the category). A prototypical example is the commutative square:

$$\begin{tikzcd} a \arrow[r, "g"] \arrow[d, "f"'] & c \arrow[d, "h"] \\ b \arrow[r, "k"'] & d \end{tikzcd}$$

In this diagram there are two linear compositions, $a \xrightarrow{f} b \xrightarrow{k} d$ and $a \xrightarrow{g} c \xrightarrow{h} d$, and when we assert that the diagram commutes, we are asserting that these two linear compositions are equal.

Pasting diagrams are similar. They consist of polygonal arrangements in the plane

with appropriate labels. Such an arrangement is a two-dimensional analogue of the linear arrangements for ordinary categories. The vertices of a pasting diagram are labeled with objects of the bicategory under consideration, the (directed) edges are labeled with 1-morphisms, and the polygonal regions themselves are labeled with 2-morphisms (and directions). The basic situations are the following:

$$\begin{tikzcd}
a \arrow[rr, "g"] \arrow[dr, "f"'] \arrow[r, phantom, "\alpha \Downarrow"'] & & c \arrow[dr, "h"] \arrow[d, phantom, "\beta \Downarrow"] & \\
& b \arrow[ur, "l"] \arrow[rr, "k"'] & & d
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& a \arrow[rr, "g"] \arrow[dr, "l"] & & c \\
b \arrow[ur, "f"] \arrow[rr, "k"'] \arrow[r, phantom, "\alpha \Downarrow"] & & d \arrow[ur, "h"'] \arrow[u, phantom, "\beta \Downarrow"] &
\end{tikzcd}$$

which are to be interpreted as the compositions $(\beta * id_f) \circ (id_h * \alpha)$ and $(\alpha * id_h) \circ (id_f * \beta)$. Out of these basic composites we can give meaning to more complex diagrams such as the one in Figure B.3.

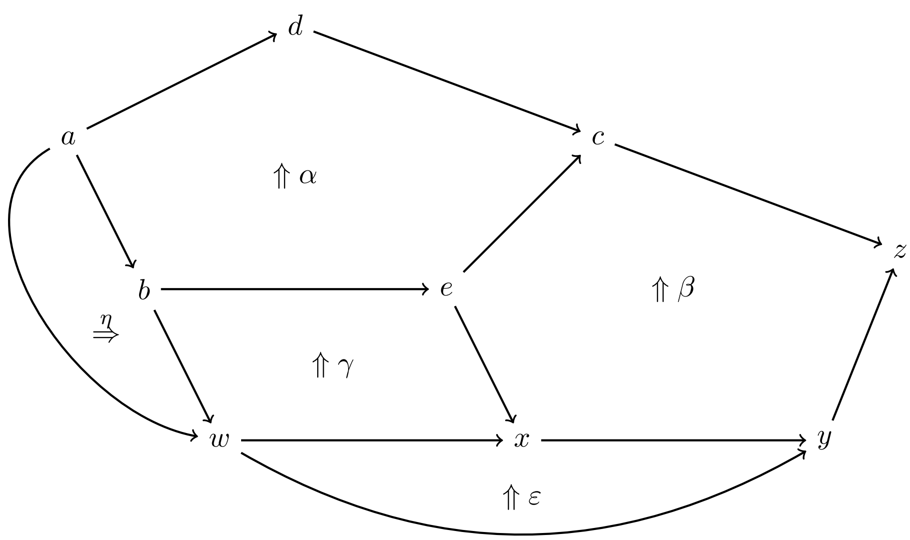


Figure B.3: An Example of a Pasting Diagram

In a 2-category, there is a unique 2-morphism specified by each such pasting diagram. This is not the case in general bicategories. Such a diagram is supposed to represent the vertical composition of 2-morphisms which are horizontal compositions like:

$$\begin{tikzcd}
 & & e \arrow[dr, "j"] & & \\
a \arrow[r, "f"] & b \arrow[ur, "g"] \arrow[dr, "h"'] \arrow[rr, phantom, "\Uparrow\gamma"] & & x \arrow[r, "k"] & y \\
 & & w \arrow[ur, "i"'] & &
\end{tikzcd}$$

but to make sense of such horizontal compositions, one needs to choose a bracketing. Does the above picture mean $(id_k * \gamma) * id_f$ or $id_k * (\gamma * id_f)$?

Worse, to make sense of the entire pasting diagram we must also change the bracketing *en route*. The above horizontal composition is a 2-morphism which starts at (say) $(k \circ (i \circ h)) \circ f$, but in the pasting diagram in Figure B.3 we must compose it with $\varepsilon * \eta$, which ends at $(k \circ i) \circ (h \circ f)$. In a general bicategory these compositions might not agree and so the 2-morphisms may not be composable.

Nevertheless, in any bicategory MacLane's Coherence Theorem (Theorem B.2.1) ensures that there are canonical coherence isomorphisms between rebracketed expsessions. So for example, we have a canonical isomorphism $(k \circ i) \circ (h \circ f) \cong (k \circ (i \circ h)) \circ f$. When we draw a pasting diagram we will implicitly insert these canonical coherence isomorphisms. With this convention, a pasting diagram in an arbitrary bicategory takes a unique value, provided we specify a bracketing of the *outside* 1-morphisms. This can be proven by an appropriate induction on polygonal decompositions of the disk.

When we write equations of the form "pasting diagram A = pasting diagram B", what we mean is that when we give these pasting diagrams a fixed bracketing on the boundary, then they agree. If this is the case, then they also agree with any other bracketing on the boundary, so that nothing is lost by omitting this bracketing. The property "A = B" doesn't depend on the bracketing of the relevant pasting diagram, so we may as well omit it from the notation.

This becomes slightly problematic when we want to specify *structure* for a bicategory. For example, a monoidal bicategory is similar to a monoidal category. Both have $\otimes$-structures, and both have associators α. But while these structures for a monoidal category satisfy the pentagon identity, in a monoidal bicategory this identity becomes an extra piece of data: the pentagonator. This is a 2-isomorphism which is supposed to fill diagrams such as:

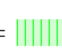
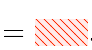
$$\begin{tikzcd}
 & (a\otimes b)\otimes(c\otimes d) \arrow[dr, "\alpha"] & \\
((a\otimes b)\otimes c)\otimes d \arrow[ur, "\alpha"] \arrow[d, "\alpha\otimes I"'] \arrow[rr, phantom, "\Uparrow \pi"] & & a\otimes(b\otimes(c\otimes d)) \\
(a\otimes(b\otimes c))\otimes d \arrow[rr, bend right, "\alpha"'] & & a\otimes((b\otimes c)\otimes d) \arrow[u, "I\otimes\alpha"']
\end{tikzcd}$$

For this to strictly make sense, we must first bracket the bottom sequence of 1-morphisms.

Alternatively, we can interpret the pentagonator as a *family* of 2-isomorphisms, one for each possible way of bracketing the boundary (in this case two), but require that this family is *coherent*, i.e. when we change the bracketing (via MacLane's canonical coherence isomorphisms) of one member of this family, we get another member of this family. This is the point of view we will adopt here. So for example the data:

$$\begin{tikzcd}
 & a\otimes(b\otimes c) \arrow[r, "\beta"] & (b\otimes c)\otimes a \arrow[dr, "\alpha"] & \\
(a\otimes b)\otimes c \arrow[ur, "\alpha"] \arrow[dr, "\beta\otimes I"'] \arrow[rrr, phantom, "\Downarrow R"] & & & b\otimes(c\otimes a) \\
 & (b\otimes a)\otimes c \arrow[r, "\alpha"'] & b\otimes(a\otimes c) \arrow[ur, "I\otimes\beta"'] &
\end{tikzcd}$$

represents a family of four 2-morphisms, one for each of the four ways of bracketing the boundary.

String diagrams are an equivalent two-dimensional notation which is in some sense dual to pasting diagrams. In a string diagram an object is represented, not as a vertex, but as a *region*. A 1-morphism is drawn as a boundary between regions, and a 2-morphism is drawn as a vertex where several 1-morphisms meet. Usually these vertices are drawn as nodes, with the name of the 2-morphism labeling the node. Figure B.4 shows how a typical 2-morphism would be encoded in both the pasting diagram notation and the string diagram notation. Notice that in the string diagram we have labeled the regions corresponding to different objects with different colors; we have $a =$ , $b =$ , $c =$, and $d =$ .

$$\begin{tikzcd} & b \arrow[dr, "g"] & \\ a \arrow[ur, "f"] \arrow[dr, "f'"'] \arrow[rr, phantom, "\Downarrow u"] & & c \\ & d \arrow[ur, "g'"'] & \end{tikzcd}$$

vs

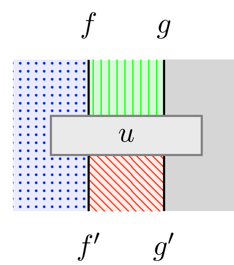


Figure B.4: Pasting Diagrams vs String Diagrams

Figure B.5 shows the string diagram which is equivalent to the pasting diagram of Figure B.3. Unless otherwise stated, we will read our string diagrams from left to right and from top to bottom, as in both these Figures.

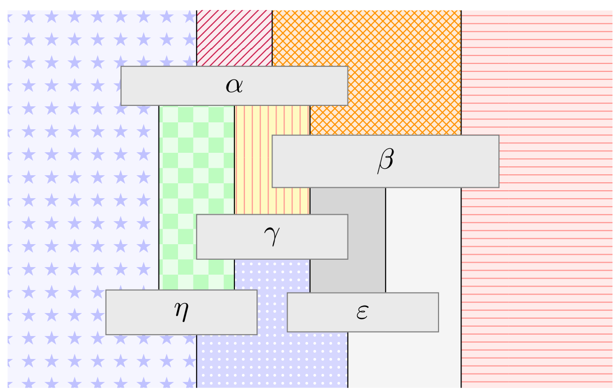


Figure B.5: An Example of a String Diagram

Duality in bicategories (also called adjunction) is an important concept which we will now turn to. String diagrams are particularly well suited for describing duality in bicategories, as we shall see shortly.

Definition B.4.1. An *adjunction* in a bicategory B is a quadruple $(f, g, \eta, \varepsilon)$ where $f : a \to b$ and $g : b \to a$ are 1-morphisms in B, $\eta : I_a \to g \circ f$ and $\varepsilon : f \circ g \to I_b$ are 2-morphisms, such that the composites:

$$f \xrightarrow{r^{-1}} f \circ I_a \xrightarrow{id * \eta} f \circ (g \circ f) \xrightarrow{a^{-1}} (f \circ g) \circ f \xrightarrow{\varepsilon * id} I_b \circ f \xrightarrow{\ell} f$$

$$g \xrightarrow{\ell^{-1}} I_a \circ g \xrightarrow{\eta * id} (g \circ f) \circ g \xrightarrow{a} g \circ (f \circ g) \xrightarrow{id * \varepsilon} g \circ I_b \xrightarrow{r} g$$

are identities. We say that f is *left-adjoint* (or *left-dual*) to g and that g is *right-adjoint* (or *right-dual*) to f. We will also denote an adjunction by $f : a \rightleftarrows b : g$ ⋄

When the bicategory $\mathsf{B} = \mathsf{Cat}$, the bicategory of categories, functors and natural transformations, then this is equivalent to the usual notion of adjoint functor, see [Mac71]. We can translate this definition into string diagrams as follows. The 2-morphisms η and ε can be described by string diagrams as in Figure B.6, and then the above equations become the equations of Figure B.7, justifying our choice of graphical depiction.

Figure B.6: Duality via String Diagrams

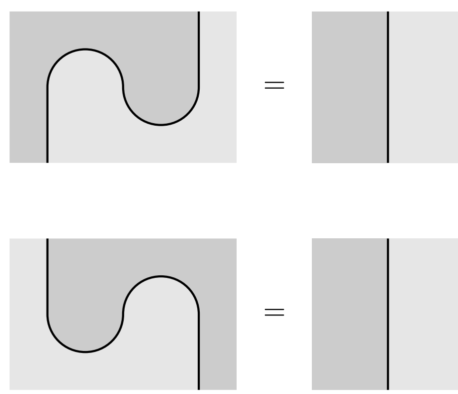


Figure B.7: Adjunction Equations via String Diagrams

Definition B.4.2. An *adjoint equivalence* is an adjunction $(f, g, \eta, \varepsilon)$ in which the 2-morphisms η and ε are invertible. ⋄

If f is left adjoint to g, with adjunction data $(f, g, \eta, \varepsilon)$, and this adjunction is an adjoint equivalence, then $(g, f, \varepsilon^{-1}, \eta^{-1})$ is an adjunction realizing f as a right adjoint to g.

Proposition B.4.3. *The following notions are logically equivalent for any 1-morphism $f : a \to b$ in a bicategory:*

1. *f is an equivalence.*

2. *f is part of an adjoint equivalence $(f, g, \eta, \varepsilon)$.*

Proof. The implication (2) $\Rightarrow$ (1) is trivial. For the reverse implication, suppose that f is an equivalence so that there exists $g : b \to a$ and isomorphisms $\eta : I_a \to g \circ f$ and $\overline{\varepsilon} : f \circ g \to I_b$. If these satisfy the adjunction equations, then we are done. Otherwise the composition,

$$f \xrightarrow{r^{-1}} f \circ I_a \xrightarrow{id * \eta} f \circ (g \circ f) \xrightarrow{a^{-1}} (f \circ g) \circ f \xrightarrow{\varepsilon * id} I_b \circ f \xrightarrow{\ell} f$$

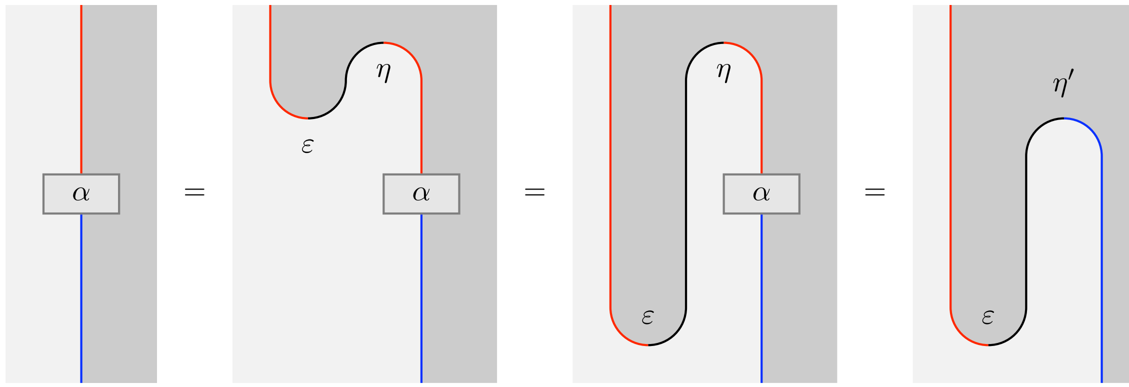
defines a non-identity isomorphism of f. Call this isomorphism α. Let $\varepsilon = \overline{\varepsilon} \circ (\alpha^{-1} * id_g)$. Then $\varepsilon : f \circ g \to I_b$ is isomorphism and the data $(f, g, \eta, \varepsilon)$ forms the desired adjoint equivalence. $\square$

Fix a 1-morphism $f : a \to b$ in B. The adjunctions $(f, g, \eta, \varepsilon)$ in B become a category where a morphism from $(f, g, \eta, \varepsilon)$ to $(f, g', \eta', \varepsilon')$ is a 2-morphism $\alpha : g \to g'$ such that $(\alpha * id_f) \circ \eta = \eta'$.

Definition B.4.4. Let 1 denote a singleton category. A category C is *contractible* if the unique functor $C \to 1$ is an equivalence. $\diamond$

Proposition B.4.5. *Let $f : a \to b$ be a 1-morphism. The category of adjunction data (f, g, η, ϵ) is either empty or a contractible category.*

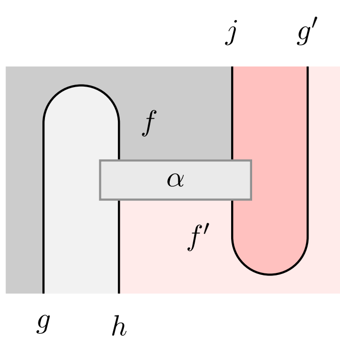
Proof. We must show that between any two adjunctions, $(f, g, \eta, \varepsilon)$ and $(f, g', \eta', \varepsilon')$, there is a unique morphism. Let $\alpha : g \to g'$ be a morphism of adjunctions. Then the following string diagrams are equal:

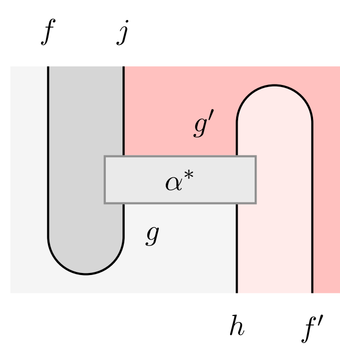


The last equality follows from the fact that α is a morphism of adjunctions. This proves uniqueness, since any two morphisms must in fact be equal to the morphism represented by the last string diagram. Moreover it is straightforward to check that the last string diagram defines a 2-morphism which is a morphism of adjunctions, proving existence. □

Proposition B.4.6 ("Mates" [KS74])**.** *Let $f : a \rightleftarrows b : g$ and $f' : a' \rightleftarrows b' : g'$ be adjunctions in a bicategory. Let $h : a \to a'$ and $j : b \to b'$ be 1-morphisms. Then there is a bijection between 2-morphisms $\alpha : j \circ f \to f' \circ h$ and $\alpha^* : g' \circ j \to h \circ g$.*

Proof. We will translate the proof given in [KS74] into the language of string diagrams. Given α or α^*, we construct its partner as the 2-morphism given by one of the following pasting diagrams:

$\alpha^* =$ 

$\alpha =$ 

These are easily seen to be inverse correspondences. □

Under this correspondence we say α^* is the *mate* of α, and vice versa. Essentially we have used the adjunction data to alter the source and target of one kind of 2-morphism to be the source and target of another kind of 2-morphism. In an analogous manner, we may use adjoint 1-morphisms to alter the source and target of 2-morphisms with more

complicated source and target arrangements. We will also refer to these more elaborate alterations as *mates* of the original 2-morphism. We will use this primarily in the situation where the adjunctions in question are adjoint equivalences.

Appendix C

Cartesian Arrows, Fibered Categories and Symmetric Monoidal Stacks

C.1 Sites and Sheaves

Definition C.1.1. A *coverage* on a category $\mathcal{D}$ is a function assigning to each object $D \in \mathcal{D}$ a collection $\tau(D)$ of families $\{f_i : D_i \to D\}$ called *covering families*, satisfying the following properties:

1. If $f : C \to D$ is an isomorphism, then $\{f : C \to D\}$ is a covering family.

2. Given a covering family $\{D_i \to D\}$ and a morphism $C \to D$, the fiber products $C \times_D D_i$ exist in D and the family $\{C \times_D D_i \to C\}$ is a covering family.

3. If $\{f_i : D_i \to D\}$ is a covering family and is for each i, one has a covering family $\{g_{ij} : D_{ij} \to D_i\}$, then the family of composites $\{f_i g_{ij} : D_{ij} \to D\}$ is a covering family.

If a covering family consists of a single map $\{f : C \to D\}$, then we call $f : C \to D$ a *covering map*. We often abuse notion and call C a *cover* of D. A coverage is also called a *Grothendieck pretopology* and also sometimes a *Grothendieck topology*. $\diamond$

Definition C.1.2. A *site* is a category equipped with a coverage. $\diamond$

Definition C.1.3. Let $\mathcal{D}$ be a site and $F : \mathcal{D}^{op} \to \mathsf{Set}$ a presheaf. F is a *sheaf* if for each covering family $\{D_i \to D\}$, the diagram,

$$F(D) \to \prod_i F(D_i) \rightrightarrows \prod_{ij} F(D_i \times_D D_j)$$

is an equalizer. $\diamond$

C.2 Cartesian Arrows and Fibered Categories

In this section we study categories over a fixed category $\mathcal{C}$, that is categories $\mathcal{F}$ equipped with a functor $p : \mathcal{F} \to \mathcal{C}$. Recall the following well known definition.

Definition C.2.1 ([Vis05])**.** Let $\mathcal{F}$ be a category over $\mathcal{C}$. An arrow $\phi : \xi \to \eta$ of $\mathcal{F}$ is *cartesian* if for any arrow $\psi : \zeta \to \eta$ in $\mathcal{F}$ and any arrow $h : p\zeta \to p\xi$ in $\mathcal{C}$ with $p\phi \circ h = p\psi$, there exists a unique arrow $\theta : \zeta \to \xi$ with $p\theta = h$ and $\phi \circ \theta = \psi$, as in the commutative diagram,

$$\begin{tikzcd}
\zeta \arrow[dr, dashed, "\theta"'] \arrow[drr, bend left, "\psi"] \arrow[dd, mapsto] & & \\
& \xi \arrow[r, "\phi"'] \arrow[dd, mapsto] & \eta \arrow[dd, mapsto] \\
p\zeta \arrow[dr, "h"'] \arrow[drr, bend left] & & \\
& p\xi \arrow[r] & p\eta
\end{tikzcd}$$

If $\xi \to \eta$ is a cartesian arrow of $\mathcal{F}$ mapping to an arrow $U \to V$ of $\mathcal{C}$, we say that ξ is a *pullback* of η to U. $\diamond$

Definition C.2.2. A *fibered category over* $\mathcal{C}$ is a category $\mathcal{F}$ over $\mathcal{C}$, such that given an arrow $f : U \to V$ in $\mathcal{C}$, and an object η of $\mathcal{F}$ mapping to V, there is a cartesian arrow $\phi : \xi \to \eta$ with $p\phi = f$. $\diamond$

Example C.2.3 (Yoneda)**.** Fix an object $X \in \mathcal{C}$. Let $\mathcal{F} = \mathcal{C}_{/X}$, the *over category of* X, i.e. the objects of $\mathcal{F}$ consist of arrows $Y \to X$ in $\mathcal{C}$ and the morphisms consist of commutative diagrams:

$$\begin{tikzcd}
Y \arrow[rr] \arrow[dr] & & Y' \arrow[dl] \\
& X &
\end{tikzcd}$$

The forgetful functor $p : \mathcal{F} \to \mathcal{C}$, sending $(Y \to X) \mapsto Y$, makes $\mathcal{F}$ into a fibered category over $\mathcal{C}$. ◇

Example C.2.4 (Sheaves)**.** Let $\mathcal{C}$ be a site and F a sheaf on this site. Consider the category $\mathcal{F} = \mathcal{C}^F$ whose objects are pairs (M, s), with $M \in \mathcal{C}$ and $s \in F(M)$, and whose morphisms from (M, s) to (M', s') are those morphisms $f : M \to M'$ of $\mathcal{C}$, such that $s = f^*(s')$. Then the forgetful functor $\mathcal{F} \to \mathcal{C}$ makes $\mathcal{F}$ into a fibered category. ◇

Given a fibered category $p : \mathcal{F} \to \mathcal{C}$ and an object $M \in \mathcal{C}$, define the category $\mathcal{F}(M)$ to consist of all those objects of $\mathcal{F}$ which map via p to M, and all those morphisms in $\mathcal{F}$ which map via p to $id_M : M \to M$. The reader may check that this is, indeed, a category. In the above two examples, $\mathcal{C}_{/X}(Y) = \mathcal{C}(Y, X)$ and $\mathcal{C}^F(Y) = F(Y)$ are discrete categories with only identity arrows.

Example C.2.5 (Principal Bundles)**.** Let $\mathcal{C} = \mathsf{Man}^d$ be the site of d-dimensional manifolds with inclusions as morphisms. Let G be a Lie group, and let $\mathcal{F} = \mathsf{Prin}_G$ be the category whose objects consist of a d-manifold M, together with a G-principal bundle over M. The morphisms consist of pull-back diagrams,

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner" very near start] & P' \arrow[d] \\ M \arrow[r, hook] & M' \end{tikzcd}$$

Then the forgetful functor $\mathcal{F} \to \mathsf{Man}^d$, makes $\mathcal{F}$ into a fibered category. $\mathcal{F}(M) = \mathsf{Prin}_G(M)$, the category of G-principal bundles over M. ◇

Example C.2.6 (Orientations)**.** Again let $\mathcal{C} = \mathsf{Man}_d$ be the site of d-manifolds. Let Or_d be the category of oriented d-manifolds with oriented inclusions as morphisms. The Forgetful functor $\mathsf{Or}_d \to \mathsf{Man}_d$ makes Or_d into a fibered category. $\mathsf{Or}_d(M)$ consists of the discrete category of the set of orientations on M. There are no non-identity morphisms in $\mathsf{Or}_d(M)$. ◇

Definition C.2.7. Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal category. A *symmetric monoidal fibered category* over $\mathcal{C}$ is a symmetric monoidal category $\mathcal{F}$, with a symmetric monoidal functor $p : \mathcal{F} \to \mathcal{C}$, such that p makes $\mathcal{F}$ into a fibered category over $\mathcal{C}$. ◇

C.3 Symmetric Monoidal Stacks

A stack is a fibered category over a site which satisfies a gluing condition reminiscent of a sheaf. Let $\mathcal{D}$ be a site, and let $\{D_i \to D\}$ be a covering family. Let $\mathcal{F} \to \mathcal{D}$ be a fibered category. We will consider two associated categories. Consider the following commutative cube of pullback diagrams in $\mathcal{D}$,

$$\begin{tikzcd}
D_{ijk} \arrow[rr] \arrow[dr] \arrow[dd] & & D_{jk} \arrow[dr] \arrow[dd] & \\
& D_{ik} \arrow[rr] \arrow[dd] & & D_k \arrow[dd] \\
D_{ij} \arrow[rr] \arrow[dr] & & D_j \arrow[dr] & \\
& D_i \arrow[rr] & & D
\end{tikzcd}$$

Here i, j, k range over the index set of the cover and $D_{ij} = D_i \times_D D_j$, and similarly for D_{ijk}.

Let $\mathcal{F}_{\mathrm{comp}}(D_i \to D)$ denote the category of commutative cubes in $\mathcal{F}$,

$$\begin{tikzcd}
\xi_{ijk} \arrow[rr] \arrow[dr] \arrow[dd] & & \xi_{jk} \arrow[dr] \arrow[dd] & \\
& \xi_{ik} \arrow[rr] \arrow[dd] & & \xi_k \arrow[dd] \\
\xi_{ij} \arrow[rr] \arrow[dr] & & \xi_j \arrow[dr] & \\
& \xi_i \arrow[rr] & & \xi
\end{tikzcd}$$

in which every arrow is cartesian and whose image in $\mathcal{D}$ is the previous cube. There is an obvious functor $\mathcal{F}_{\mathrm{comp}}(D_i \to D) \to \mathcal{F}(D)$, given by sending the cube $\{\xi_\alpha\}$ to the single object ξ. Since $\mathcal{F}$ is a fibered category, this is an equivalence of categories.

We can make a similar category $\mathcal{F}_{\mathrm{desc}}(D_i \to D)$, by omitting the object ξ from the cube data. Thus an object of $\mathcal{F}_{\mathrm{desc}}(D_i \to D)$, consists of a commutative diagram,

$$\begin{tikzcd}
\xi_{ijk} \arrow[rr] \arrow[dr] \arrow[dd] & & \xi_{jk} \arrow[dr] \arrow[dd] & \\
& \xi_{ik} \arrow[rr] \arrow[dd] & & \xi_k \\
\xi_{ij} \arrow[rr] \arrow[dr] & & \xi_j & \\
& \xi_i & &
\end{tikzcd}$$

in $\mathcal{F}$ in which every arrow is cartesian and whose image in $\mathcal{D}$ agrees with the appropriate portion of the original pullback cube. Again there is an obvious functor $\mathcal{F}_{\text{comp}}(D_i \to D) \to \mathcal{F}_{\text{desc}}(D_i \to D)$, given by forgetting the object ξ.

Definition C.3.1. Let $\mathcal{F} \to \mathcal{D}$ be a fibered category over a site. Then $\mathcal{F}$ is a *stack* if the functor

$$\mathcal{F}_{\text{comp}}(D_i \to D) \to \mathcal{F}_{\text{desc}}(D_i \to D)$$

is an equivalence of categories for each covering family $\{D_i \to D\}$. A *symmetric monoidal stack* is a symmetric monoidal fibered category $\mathcal{F} \to \mathcal{D}$, which is also a stack. $\diamond$

Remark C.3.2. All the examples defined over Man_d in the previous section are symmetric monoidal stacks. $\diamond$

Bibliography

[Abr96] Lowell Abrams. Two-dimensional topological quantum field theories and Frobenius algebras. *J. Knot Theory Ramifications*, 5(5):569–587, 1996.

[Ati88] Michael Atiyah. Topological quantum field theories. *Inst. Hautes Études Sci. Publ. Math.*, (68):175–186 (1989), 1988.

[Bén67] Jean Bénabou. Introduction to bicategories. In *Reports of the Midwest Category Seminar*, pages 1–77. Springer, Berlin, 1967.

[BN96] John C. Baez and Martin Neuchl. Higher-dimensional algebra. I. Braided monoidal 2-categories. *Adv. Math.*, 121(2):196–244, 1996. arXiv: 9511.5153.

[Cer70] Jean Cerf. La stratification naturelle des espaces de fonctions différentiables réelles et le théorème de la pseudo-isotopie. *Inst. Hautes Études Sci. Publ. Math.*, (39):5–173, 1970.

[CFW08] Danny Calegari, Michael Freedman, and Kevin Walker. Positivity of the universal pairing in 3 dimensions. 2008. arXiv: 0802.3208.

[CL04] Eugenia Cheng and Aaron Lauda. *Higher-Dimensional Categories: An Illustrated Guide Book*. 2004. preprint.

[DI71] Frank DeMeyer and Edward Ingraham. *Separable algebras over commutative rings*. Lecture Notes in Mathematics, Vol. 181. Springer-Verlag, Berlin, 1971.

[Dij89] R. Dijkgraaf. A geometrical approach to two dimensional conformal field theory. 1989. Ph.D. thesis (Utrecht).

[DS97] Brian Day and Ross Street. Monoidal bicategories and Hopf algebroids. *Adv. Math.*, 129(1):99–157, 1997.

[DSP] Christopher L. Douglas and Christopher Schommer-Pries. in preparation.

[Dub96] Boris Dubrovin. Geometry of 2D topological field theories. In *Integrable systems and quantum groups (Montecatini Terme, 1993)*, volume 1620 of *Lecture Notes in Math.*, pages 120–348. Springer, Berlin, 1996.

[Ehr51a] Charles Ehresmann. Les prolongements d'une variété différentiable. I. Calcul des jets, prolongement principal. *C. R. Acad. Sci. Paris*, 233:598–600, 1951.

[Ehr51b] Charles Ehresmann. Les prolongements d'une variété différentiable. II. L'espace des jets d'ordre r de V_n dans V_m. *C. R. Acad. Sci. Paris*, 233:777–779, 1951.

[Ehr51c] Charles Ehresmann. Les prolongements d'une variété différentiable. III. Transitivité des prolongements. *C. R. Acad. Sci. Paris*, 233:1081–1083, 1951.

[FHLT09] Daniel Freed, Michael Hopkins, Jacob Lurie, and Constantin Teleman. Topological quantum field theories from compact lie groups. 2009. arXiv: 0905.0731.

[FKN+05] Michael H. Freedman, Alexei Kitaev, Chetan Nayak, Johannes K. Slingerland, Kevin Walker, and Zhenghan Wang. Universal manifold pairings and positivity. *Geom. Topol.*, 9:2303–2317 (electronic), 2005.

[GG73] M. Golubitsky and V. Guillemin. *Stable mappings and their singularities.* Springer-Verlag, New York, 1973. Graduate Texts in Mathematics, Vol. 14.

[GPS95] R. Gordon, A. J. Power, and Ross Street. Coherence for tricategories. *Mem. Amer. Math. Soc.*, 117(558):vi+81, 1995.

[HKST08] Henning Hohnhold, Mathias Kreck, Stephan Stolz, and Peter Teichner. Super symmetric euclidean field theories and generalized cohomology, a survey. 2008. preprint.

[Izu00] Masaki Izumi. The structure of sectors associated with Longo-Rehren inclusions. I. General theory. *Comm. Math. Phys.*, 213(1):127–179, 2000.

[Jän68] Klaus Jänich. On the classification of $O(n)$-manifolds. *Math. Ann.*, 176:53–76, 1968.

[JS86] André Joyal and Ross Street. Braided tensor categories. *Macquarie Mathematics Report*, (860081), 1986.

[KL01] Thomas Kerler and Volodymyr V. Lyubashenko. *Non-semisimple topological quantum field theories for 3-manifolds with corners*, volume 1765 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2001.

[KMS93] Ivan Kolář, Peter W. Michor, and Jan Slovák. *Natural operations in differential geometry*. Springer-Verlag, Berlin, 1993.

[Koc04] Joachim Kock. *Frobenius algebras and 2D topological quantum field theories*, volume 59 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2004.

[KS74] G. M. Kelly and Ross Street. Review of the elements of 2-categories. In *Category Seminar (Proc. Sem., Sydney, 1972/1973)*, pages 75–103. Lecture Notes in Math., Vol. 420. Springer, Berlin, 1974.

[KT08] Matthias Kreck and Peter Teichner. Positive topological field theories and manifolds of dimension ¿ 4. 2008. Preprint. To appear in Journal of Topology, Volume 1 (2008), 663–670,.

[KV94a] M. Kapranov and V. Voevodsky. Braided monoidal 2-categories and Manin-Schechtman higher braid groups. *J. Pure Appl. Algebra*, 92(3):241–267, 1994.

[KV94b] M. M. Kapranov and V. A. Voevodsky. 2-categories and Zamolodchikov tetrahedra equations. In *Algebraic groups and their generalizations: quantum and infinite-dimensional methods (University Park, PA, 1991)*, volume 56 of *Proc. Sympos. Pure Math.*, pages 177–259. Amer. Math. Soc., Providence, RI, 1994.

[Lam99] T. Y. Lam. *Lectures on modules and rings*, volume 189 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1999.

[Lau00] Gerd Laures. On cobordism of manifolds with corners. *Trans. Amer. Math. Soc.*, 352(12):5667–5688 (electronic), 2000.

[Lei02] Tom Leinster. A survey of definitions of n-category. *Theory Appl. Categ.*, 10:1–70 (electronic), 2002.

[Lei04] Tom Leinster. *Higher operads, higher categories*, volume 298 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2004.

[Lur09] Jacob Lurie. On the classification of topological field theories. 2009. preprint.

[Mac71] Saunders MacLane. *Categories for the working mathematician*. Springer-Verlag, New York, 1971. Graduate Texts in Mathematics, Vol. 5.

[McC00] Paddy McCrudden. Balanced coalgebroids. *Theory Appl. Categ.*, 7:No. 6, 71–147 (electronic), 2000.

[Mic80] Peter W. Michor. *Manifolds of differentiable mappings*, volume 3 of *Shiva Mathematics Series*. Shiva Publishing Ltd., Nantwich, 1980.

[Mor07] Jeffrey Morton. Extended TQFT's and quantum gravity. 2007. Ph.D Thesis (University of California, Riverside) arXiv:0710.0032.

[MS06] G. Moore and G. Segal. D-branes and K-theory in 2D topological field theory. 2006. arXiv:0609.3042.

[Mun66] James R. Munkres. *Elementary differential topology*, volume 1961 of *Lectures given at Massachusetts Institute of Technology, Fall*. Princeton University Press, Princeton, N.J., 1966.

[Por71] I. R. Porteous. Simple singularities of maps. In *Proceedings of Liverpool Singularities Symposium, I (1969/70)*, pages 286–307. Lecture Notes in Math., Vol. 192, Berlin, 1971. Springer.

[Por72] Ian R. Porteous. The second-order decomposition of Σ^2. *Topology*, 11:325–334, 1972.

[Qui95] Frank Quinn. Lectures on axiomatic topological quantum field theory. In *Geometry and quantum field theory (Park City, UT, 1991)*, volume 1 of *IAS/Park City Math. Ser.*, pages 323–453. Amer. Math. Soc., Providence, RI, 1995.

[Saw95] Stephen Sawin. Direct sum decompositions and indecomposable TQFTs. *J. Math. Phys.*, 36(12):6673–6680, 1995. arXiv:q-alg/9505026v1.

[Seg04] Graeme Segal. The definition of conformal field theory. In *Topology, geometry and quantum field theory*, volume 308 of *London Math. Soc. Lecture Note Ser.*, pages 421–577. Cambridge Univ. Press, Cambridge, 2004.

[ST04] Stephan Stolz and Peter Teichner. What is an elliptic object? In *Topology, geometry and quantum field theory*, volume 308 of *London Math. Soc. Lecture Note Ser.*, pages 247–343. Cambridge Univ. Press, Cambridge, 2004.

[ST08] Stephan Stolz and Peter Teichner. Super symmetric euclidean field theories and generalized cohomology, a survey. 2008. preprint.

[Vis05] Angelo Vistoli. Grothendieck topologies, fibered categories and descent theory. In *Fundamental algebraic geometry*, volume 123 of *Math. Surveys Monogr.*, pages 1–104. Amer. Math. Soc., Providence, RI, 2005.

[Vor94] A. A. Voronov. Topological field theories, string backgrounds and homotopy algebras. *Adv. Appl. Clifford Algebras*, 4(Suppl. 1):167–178, 1994. Differential geometric methods in theoretical physics (Ixtapa-Zihuatanejo, 1993).